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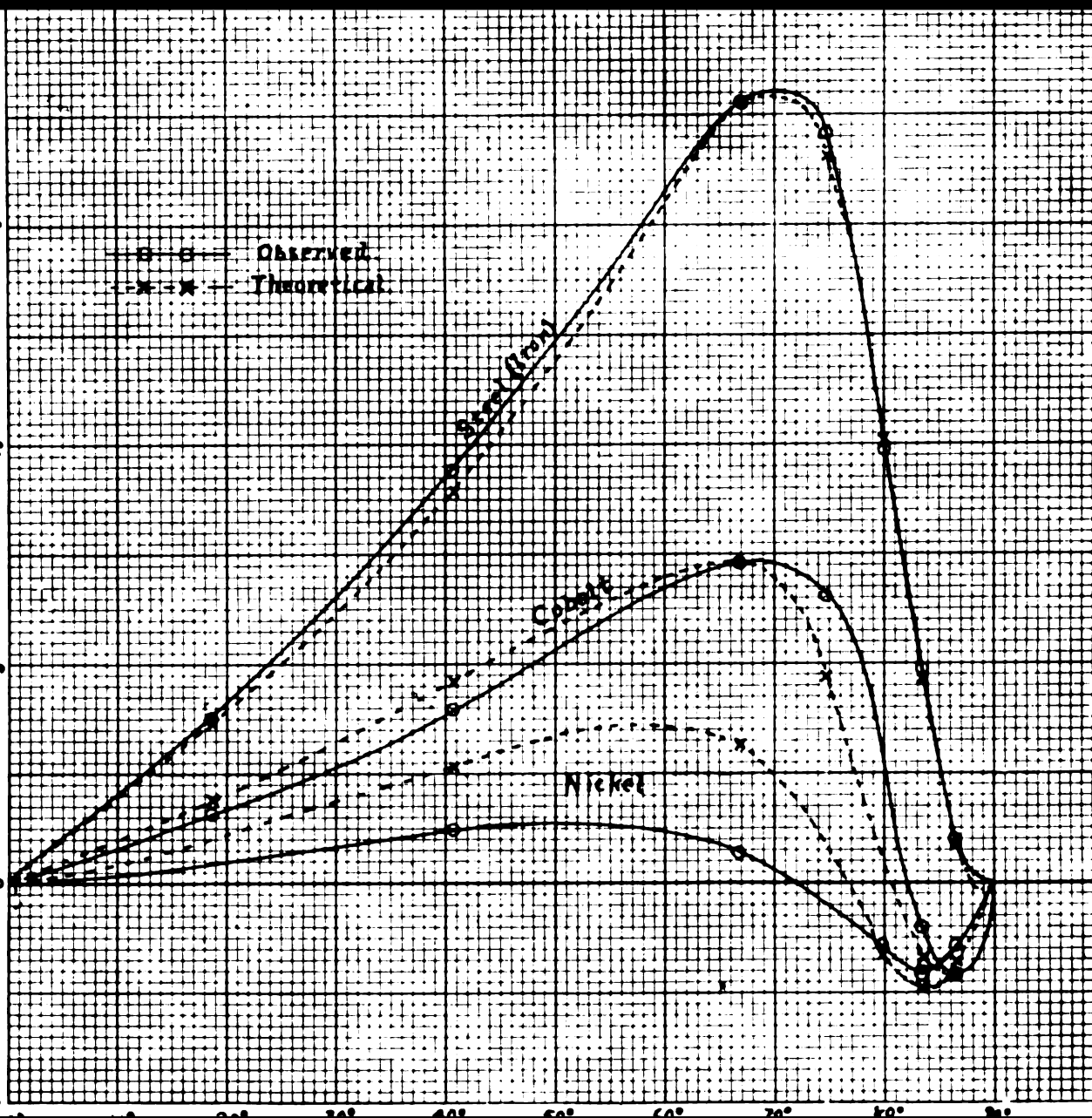
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THE
PHYSICAL REVIEW

*A JOURNAL OF EXPERIMENTAL AND
THEORETICAL PHYSICS*

CONDUCTED WITH THE CO-OPERATION
OF THE
AMERICAN PHYSICAL SOCIETY
BY
EDWARD L. NICHOLS, ERNEST MERRITT,
AND FREDERICK BEDELL

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THE PHYSICAL REVIEW.

ON THE VIBRATION OF A LECHER SYSTEM USING A LECHER OSCILLATOR, II.¹

BY F. C. BLAKE AND CHARLES SHEARD.

IN our former paper² we described some experiments performed on a Lecher system using a Paalzow-Rubens bolometer. The present paper is a continuation of that work but instead of a bolometer a thermal couple was employed. This has the advantage over the bolometer of greater constancy, because it is not dependent upon any auxiliary battery circuit. Its introduction led to some rather interesting results as will be seen further along in this paper.

APPARATUS.

The apparatus, Fig. 1, was so mounted on a system of ways that the oscillator and Lecher circuits could each be bodily moved along the ways parallel to its own length. The end-capacity plates, eight in all, were made of tin 5 cm. in diameter and 0.3 mm. thick. The total length of the main receiver was roughly that of the bolometer circuit previously used. This system of ways afforded a ready means of changing the coupling at either end of the Lecher circuit. The precautions taken in the first paper for an uncontaminated wave-system were continued here, of course. Indeed, the two precautions used by Blake and Ruppertsberg³ of preventing stray radiation effects, though not used in our earlier paper, were again employed in the present experiments. The thermal couple was enclosed in a large metal box $2 \times 1\frac{1}{2} \times 1\frac{1}{2}$ ft., the lead wires being surrounded with 10-inch furnace piping up to within a foot of the end-capacity plates. The wires leading to the galvanometer were encased in an iron pipe, the pipe end near the galvanometer being stuffed

¹ We gratefully acknowledge our thanks to the trustees of the Elizabeth Thompson Science Fund for partial financial assistance in carrying on the work here reported.

² *Phys. Rev.*, Vol. XXXII., page 533, 1911.

³ *Phys. Rev.*, Vol. XXXII., page 449, 1911.

with tinfoil and the pipe earthed. In this way the minimum readings were cut down at times to a value as low as 0.3 per cent. of the maximum for the free fundamental. While these precautions can affect the vibrations as to *type* not at all, nevertheless they are essential for accurate work.

As the work progressed it became evident that the degree of coupling at each end of the Lecher circuit and the relative lengths of the various circuits played such an important part in the character of the vibrations that it became tedious and unnecessary to continue the check receiver readings in order to obtain satisfactory results. All of the curves here shown except those of Figs. 8 and 13 were taken with the use of a main receiver alone. They cannot, of course, be as accurate as some of the curves of our first paper (*e. g.*, those of Fig. 9). Nevertheless we feel sure the reader will agree that they cannot be essentially in error on that account.

RESONANCE BETWEEN THE OSCILLATOR AND LECHER CIRCUITS.

It was stated in the other paper that experiments were in progress where by employing a very long oscillator we hoped to free the Lecher system from the oscillator vibrations. This hope, however, was never realized except as it was realized through resonance conditions. We have worked with oscillators varying in length (expressed as the length *AB*, Fig. 1, with the wire stretched out straight) from 50 to 400 cm. and have

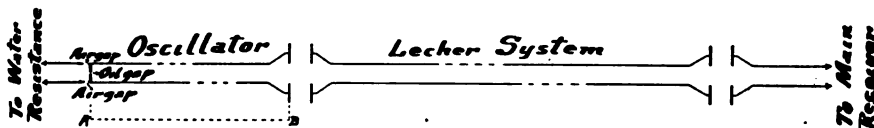


Fig. 1.

not been able to free the Lecher system from oscillator influence. Indeed it will be seen that we not only could not get rid of the oscillator influence but by substituting a thermal couple for the bolometer the receiver itself played a part in our curves. To what extent we have been able to differentiate the various effects one from another will show as the paper proceeds. Not anticipating any trouble due to the influence of the receiver upon the vibrations we started out by trying both very short and very long oscillators, but in each case the resultant set of vibrations was too complicated to be easy of explanation. We had proven that so long as the bridge length was inappreciable compared to the length of the Lecher wires, the vibrations are essentially longitudinal. Moreover, under such conditions we had shown that a circuit consisting of two

parallel wires closed at one end had possible frequency-ratios of 1, 3, 5, 7, etc., while for such wires open at both ends the possible ratios are 1, 2, 3, 4, etc. If, then, the oscillator circuit as to geometrical configuration were exactly equal to half the Lecher circuit it was figured that for a loose coupling resonance between the two circuits should obtain. When this was tried, however, it became apparent (Curve I., Fig. 2) that while we were near resonance, the oscillator was apparently too short. The data for the various circuits for all the curves here shown are collected in Table I. The oscillator 1st¹ occurs at 124.7, the free first at 119.4,

TABLE I.

Figure.	Curve Number.	Length of Oscillator.	Length of Lecher System.	Length of Receiver.	Coupling at Osc. End.	Coupling at Rec. End.
2	I.	120.0	240.0	313.5	4	4
	II.	125.5	240.0	313.5	6	6
	III.	131.2	240.0	313.5	6	6
5	I.	70.0	140.0	313.5	6	6
	II.	75.7	140.0	313.5	6	6
	III.	79.5	140.0	313.5	6	6
7	I.	90.0	169.6	313.5	6	6
	II.	92.5	169.6	313.5	6	6
8	I.	95.4	191.0	313.5	6	6
	II.	101.3	191.0	313.5	6	6
	III.	92.6	191.0	313.5	6	6
6	I.	141.0	169.6	313.5	6	6
	II.	141.0	169.6	313.5	3	3
	III.	141.0	169.6	325.5	6	6
	IV.	141.0	169.6	338.5	6	6
	V.	249.0	192.0	313.5	3	3
9	I.	97.0	186.0	313.5	6	6
	II.	102.0	186.0	313.5	6	6
	III.	97.0	186.0	313.5	6	6
(non-selective)						

giving 5.3 cm. difference. Accordingly we added 5.5 cm. to the oscillator length and obtained Curve II., Fig. 2. When we added 5.7 cm. more, we obtained Curve III. The great superiority as to type of Curve II. over Curves I. and III. will be readily admitted. In Curve II. the oscillator and Lecher circuits are exactly in resonance and every peak

¹ While trying to unravel curves similar to those of Fig. 2 we became wholly converted to the notation already used by some, of calling the harmonics of a circuit by their frequency-ratios. Accordingly in this paper, the word "fundamental" is the same as the first vibration. The first overtone of the oscillator is called "oscillator 3d," the second overtone "oscillator 5th," and so on; the first overtone of the Lecher circuit is called "free 2d," the second overtone, "free 3d," and so on. This conduces wonderfully to clearness and we strongly recommend its universal adoption. To put our previous work (l. c.) under this notation, add 1 to the figures at the top of Fig. 1a, page 535. Similarly, for Blake and Ruppertsberg's paper add 1 to the figures at the top of Fig. 4a, page 459.

total wire length being 240 cm. Now, the free thirds occur at 25.7, 118.7, 211.7 with a mean internodal space of 93.0 cm. Adding half of this to 211.7 gives 258.2; subtracting 238.7 gives 19.5 cm. as the value of the end-capacity. Subtracting 41.5 from 41.7 gives 0.2; subtracting one half of 41.7 from 0.2 gives -20.6 , the oscillator end of the Lecher wires including end-capacity. Subtracting 1.3 from 20.6 gives 19.3 as the end-capacity. Adding 41.5 to 216.5 gives 258.0, in practical agreement with 258.2 above. Subtracting 238.7 from 258 gives 19.3 as end-capacity, checking 19.5 sufficiently well.

We are not at all clear as to the mode of vibration that causes the receiver peak to appear at 91.5 cm. Now, whenever we changed the Lecher wire length the receiver wires in general were not disturbed in their length nor space-position, nor were the plates of the Lecher system at the receiver end disturbed. The zero of the meter stick, however, was always put even with the oscillator plates of the Lecher system. It was observed, then (see many of the curves in this paper), that the receiver peak *always* came at 91.5 no matter what the length of the Lecher wires, nor that of the oscillator. The fairest explanation of this fact would seem to be as follows: Adding 19.3, 1.3 and 91.5 gives 112.1 cm. This seems to be the quarter-wave-length, then, of the receiver. It will be shown later that when the receiver length is increased x cm. the receiver-peak on the Lecher wires is displaced $4/13x$ cm. toward the receiver end of the Lecher wires. Now, the total receiver length was 627 cm. roughly from plate through the thermal couple to plate. The wires from the plate to the thermal junction were practically parallel and at a distance apart of 30 cm. through the greater part of their length. The Lecher wires on the other hand were only 5 cm. apart. The receiver wires would accordingly have less capacity per unit length than the Lecher wires were it not for the fact that they were surrounded with the 10-inch furnace-piping. The extra capacity due to the piping slightly more than overcomes the decrease in capacity due to their greater distance apart. Now, $3 \times 112.1 = 336.3$ cm. while half of 627 ($= 313.5$) + 19.3 = 332.8 cm., the difference being due to the capacity differences just spoken of. If the fraction above, $4/13$, is called $1/3$, we can say that the peak at 91.5 is due to the first overtone of the receiver. This is borne out by the fact that when the Lecher wires were cut to a length so as to be in unison with the receiver wires, thus bringing this receiver peak and the free first together the tone was unusually strong; clearly the mode of vibration then must have been as in Fig. 3. Moreover, $41.5 \times 8 = 332$ cm., roughly the same as either of the numbers above. It seems then that the maxima at 9.0, 50.5, 92 (91.5), 133.5, 175, 216.5

(some of them are marked "Three") can be accounted for as a sub-multiple of the combined circuit formed from the three circuits taken together (*A*, Fig. 4), while the maxima at 41.7, 83.2, 124.7, 166.2, 207.7 (some of them are marked "Two") in the same way are due to the oscillator and Lecher circuits combined (*B*, Fig. 4). There is nothing in the nature of things against this explanation, unusual as it may seem. It



Fig. 3.

ought to be said that the greatest amount the figures have been "forced" in the one case is 0.5 cm., 92 instead of 91.5 cm., and 0.7 cm. in the other, 207.7 instead of 207.0 cm. This leaves a solitary maximum at 170 unaccounted for in Curve I., Fig. 2. It should be noted that the receiver peak at 216.5 is unusually strong, suggesting an internodal space of 3×41.5 or 124.5 cm. as shown in *A*, Fig. 4. Note also from *B*, Fig. 4, that the last node occurs 5.9 cm. behind the spark-gap. Now, this number is in good agreement with the amount that has to be added to the oscillator to make it in resonance with the Lecher system, viz., 5.5 cm.

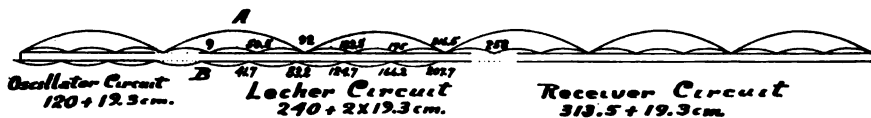


Fig. 4.

We believe that we have in Curve I., Fig. 2, an unusual opportunity to show that under certain conditions two of the circuits act together as a single circuit. Note first that the end-capacity, 19.3 cm., is roughly one half of the internodal space 41.5 cm. and that 3×41.5 is very close to 125.5 cm. This probably accounts for the fact that this internodal space is what it is. In other words, note that in *B*, Fig. 4, the potential loop spans the two capacity plates between the oscillator and Lecher circuits. (The equivalent wire-length of these two plates, 38.6 cm., is drawn as a dotted line in the figure.)

Table II. shows clearly from all three curves of Fig. 2 how from the relative displacements between the various harmonics of the oscillator and Lecher systems one can arrive at the length of the oscillator that should respond to the Lecher system. The first set of figures in column 9 show a remarkable agreement, the mean being that actually used to obtain Curve II.

TABLE II.

Figure.	Curve Number.	Frequency Number n .	Position.		Difference δ .	$\delta \times \frac{n}{n'}$.	Actual Osc. Length.	Cal. Osc. Length for Resonance.	Mean.	Osc. Length including End-capacity.	Lecher System Length including Capacity.	Oscillator "Wave" length.	Ratio.
2	I.	1	Osc.	124.7	119.4	5.3	5.3	120.0	125.3				
		5		21.0	11.0	10.0	5.6		125.6				
		5		73.0	65.0	8.0	5.7	125.5	125.7				
	II.	5		57.0	65.0	8.0	5.7	131.2	125.5				
	III.	5		170.0	173.5	3.5	5.8		125.4	125.5	278.6	579.2	2.08
5	I.	1		75.0	69.6	5.4	5.4	70.0	75.4				
		5		44.0	39.0	5.0	3.6		73.6				
		5		103.0	101.5	1.5	2.5		72.5				
	II.	3		10.0	12.0	2.0	1.2	75.7	74.5				
	III.	1		66.5	69.6	3.1	3.1	79.5	76.4				
7	II.	3		7.5	12.0	4.5	2.7		76.8	74.8	178.6	376.4	2.11
		5		40.5	45.0	4.5	3.2	92.5	89.3	89.3	208.2	434.4	2.08
		5		2.5	5.0	2.5	1.4	101.3	99.9	99.9	229.6	476.8	2.08
9	I. and III. II.	5		43.5	50.0	6.5	4.6	97.0	97.0	97.2	224.6	466.0	2.08
								102.0	97.4				

n' is the number of quarter wave-lengths between any given maximum and the receiver end of the Lecher system including end-capacity.

Considering the general features of the curves of this figure one remarks that the relative strength of the free thirds increases as the oscillator length increases through the resonance point while the reverse is true for the free 5ths. Moreover, whereas, in Curves I. and II. the free 7ths are quite strong, with the free 4ths practically absent, in Curve III. we find the 7ths practically gone with the free 4ths coming in quite strong. One is tempted to inquire whether Curve II. will not submit readily to Abraham's simple theory, but it is plain upon inspection that the free 5ths are too strong and the free 1st is too weak compared to the free 3ds to make a comparison with theory worth while. We attribute this to the receiver influence. If this could be got rid of, one should expect agreement between theory and experiment for the *odd* harmonics in the case of resonance between the oscillator and Lecher circuits. That the oscillator plays an important part can be seen by comparing the relative intensities of the even with the odd harmonics in the curves of Fig. 2.

RATIO BETWEEN THE OSCILLATOR WAVE-LENGTH AND THE LECHER WIRE-LENGTH.

We have shown clearly from the curves of Fig. 2 and from Table II. that the oscillator circuit must be longer than the half Lecher circuit in order to be in resonance with it. The ratio between the oscillator "wave-length" and the Lecher wire-length from Fig. 2 is 2.08, obtained as follows: $125.5 + 19.3$ is 144.8 cm., $240 + 2 \times 19.3 = 278.6$ cm., $(4 \times 144.8)/278.6 = 2.08$. For both circuits we have neglected the bridge length, 5 cm., for we have assumed the currents strictly longitudinal. That this is a safe assumption will be partly conceded, we think, from the results already presented. Further justification for this assumption is presented later in this paper.

Having to our surprise found the oscillator length four per cent. greater than the Lecher half-length for resonance we proceeded to try out other oscillator and Lecher lengths. Three curves similar to those of Fig. 2 are shown in Fig. 5. Tables I. and II. give the data needed to understand the curves. That Curve II. is again a better resonance curve than either I. or III. will be readily conceded. However, the free third at 12 cm. shows itself mingled with something else at 10 cm.; in other words, the oscillator was too long by a little. When it was cut down to 73.5 cm. the resonance was exact. We mention a few of the more important things about these curves. In their general features they are entirely similar to those of Fig. 2. As the oscillator is lengthened certain harmonics decrease and others increase in intensity. However, in III., Fig. 5, the oscillator shows itself very markedly, whereas in III., Fig. 2, its

maxima were relatively weak. Inspecting the free third at 127 we see how when the oscillator is too short it is pulled over to 129, although when the oscillator is too long the peak still seems at 127. However, the free 5th at 101.5 is pulled up to 103 by a short oscillator and down to 99 by a long oscillator. This sort of displacement is surely in general to be

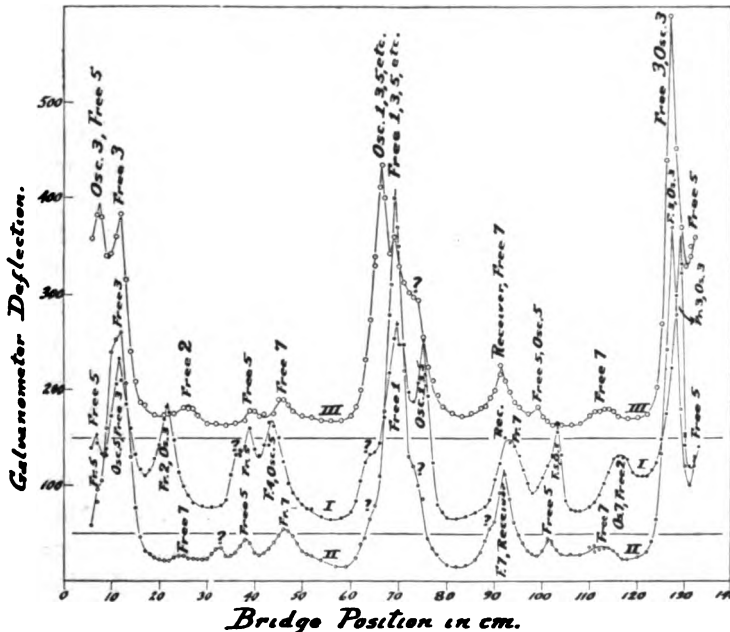


Fig. 5.

expected. We don't know that we can explain the maxima marked with an interrogation point. Apparently again some are due to the oscillator and some to the receiver. The receiver peak at 91.5 shows strongly in all three curves. It is worth remarking that peaks may be said to appear at 10.5, 37.5, 64.5, 91.5, 118.5 with an internodal space of 27 cm. Adding 27 cm. to 118.5 gives 145.5. Now for a 6 cm. coupling the end-cap is 18.6 cm. which when added to the end of the wire 138.7 gives 157.3 as the end of the wire including end capacity. Subtracting 145.5 from 157.3 gives 11.8 cm. Subtracting 11.8 from 336.3 gives 324.5 cm. Dividing this remainder by the internodal space of 27 cm. gives exactly 12. In other words the Lecher circuit and receiver circuit again behave as a single circuit. We suspect that the reason the oscillator peak at 66 is so strong in Curve III., Fig. 5, whereas in Curve III., Fig. 3, the same peak at 113 scarcely shows, is due to the presence of a receiver peak at 64.5, while in Fig. 2 the nearest receiver peak was 20 cm. away.

In Fig. 5 this leaves a peak at 33 in Curve II. and a stronger peak at 73 in both II. and III. unexplained. The fact that the latter remains stationary though the oscillator length varies points to its being some sort of a receiver peak. We will show presently that the ratio of oscillator "wave-length" to Lecher wire-length is in all cases investigated 2.08, but before we do this it is better to discuss the influence of the receiver upon our results.

INVESTIGATION OF THE RECEIVER INFLUENCE.

Having found that the receiver was exerting its own influence upon the wave system on the Lecher wires it became necessary to study its effect thereon. We first tried to see what effect a small change in length of the receiver would have. For a given receiver length the receiver peak had stood consistently at 91.5 cm. through a large number of variations of oscillator- and Lecher-lengths. The various changes we tried are recorded in Fig. 6. With an oscillator length of 141 cm., a

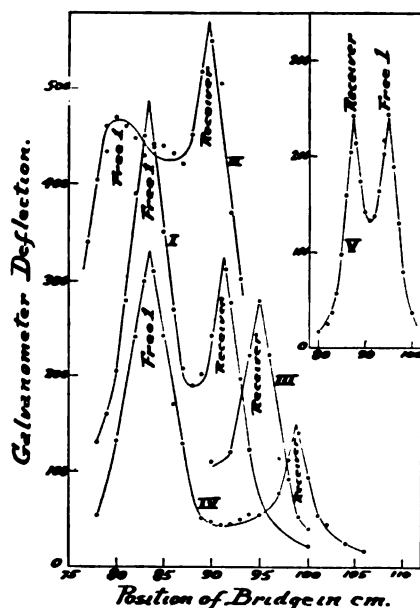


Fig. 6.

Lecher length of 170 and a coupling of 6 cm. at each end we obtained Curve I., with the free first at 83.5 cm. and the receiver peak at 91.2 cm. When the coupling was made 3 cm. at each end the receiver peak predominated over the free first, both peaks being displaced toward the oscillator because of the closer coupling, as shown in Curve II. Adding 12 cm. to each lead wire of the receiver gave Curve III. for the receiver peak with the maximum at 95 cm., coupling 6 cm. again. Adding 25 cm. to each side of the receiver changed Curve I. to Curve IV. In each of these last two cases the receiver peak is displaced $\frac{4}{13}$ of the amount added to each side

or $\frac{2}{13}$ of the total increase in receiver-length. Note that the displacement is *away* from the oscillator, and that the receiver peak gets less in intensity as the receiver length is increased. Curve V. is inserted for the following reason. Earlier in the work under different experimental conditions we had obtained Curve V. and had naturally

attributed it to the splitting up of the fundamental tone into two peaks because of close coupling (3 cm.). This was the more naturally done because the two peaks were identical in height. Later by work similar to the above we proved this was entirely wrong, one of the two peaks being due to the receiver, as marked. But it took us several weeks to run this down and to thus find out our error.

In Fig. 7, Curve I., while still investigating the receiver peak at 91 cm. we had the oscillator 90 cm. long, the Lecher wires 170 cm., coupling 6 cm. at both ends. Thinking the oscillator too short as revealed in the free third at 16 cm. we added 2.5 cm. to it, and also added 120 cm. to each side of the receiver thus getting Curve II., Fig. 7. But a study of

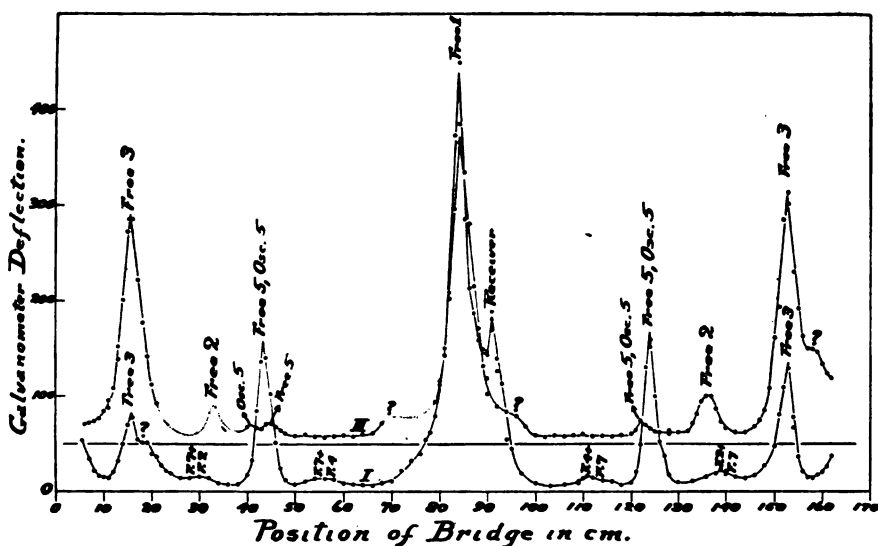


Fig. 7.

both curves (see Table II.) shows that the oscillator even in Curve I. was already too long, the maximum at 19 being due to some cause other than the oscillator 3d. A close study of the curves of this figure shows the displacing of the different maxima by the various influences able to operate. We have marked the maxima which unquestionably belong to the different harmonics even though displaced. Note that in Curve II. the receiver peak at 91 has disappeared because of the increased receiver length. From Fig. 6 this peak should have been displaced from 91 to 128 thus: $\frac{4}{13}$ of 120 = 37. $91 + 37 = 128$. But it apparently is not present. Either it has weakened so in intensity because of the great increase in length of the receiver or the ratio $\frac{4}{13}$ does not hold for the displacement. Indeed for purposes of explanation of some of the maxima

as being due to all three circuits combined swinging as a single circuit we have already (rather arbitrarily) taken this ratio as $1/3$ instead of $4/13$.

Now, early in the work, using the check receiver, we had obtained Curves I. and II. of Fig. 8 and without the use of the check receiver, Curve III. Tables I. and II. show the necessary data. At the time they were taken the peak at 91 cm. was not understood. Curve I. was taken when the oscillator circuit as to geometrical configuration was exactly equal to half of the Lecher system. It does not show the double peak for the free and oscillator fifths at 50 cm. that Curves I. of Figs. 2 and 5 and Curve II. of Fig. 7 do under similar circumstances. Indeed except for the receiver peak at 91, Curve I. is such a satisfactory curve that it would seem that the ratio between the oscillator "wave-length" and the Lecher wire-length ought to be exactly 2.00 instead of 2.08. This leads us to ask the question, May not this ratio 2.08 be due after all to the receiver influence since, when the receiver is so nearly in tune with the free first as in the curves of Fig. 8, the free maxima are apparently the

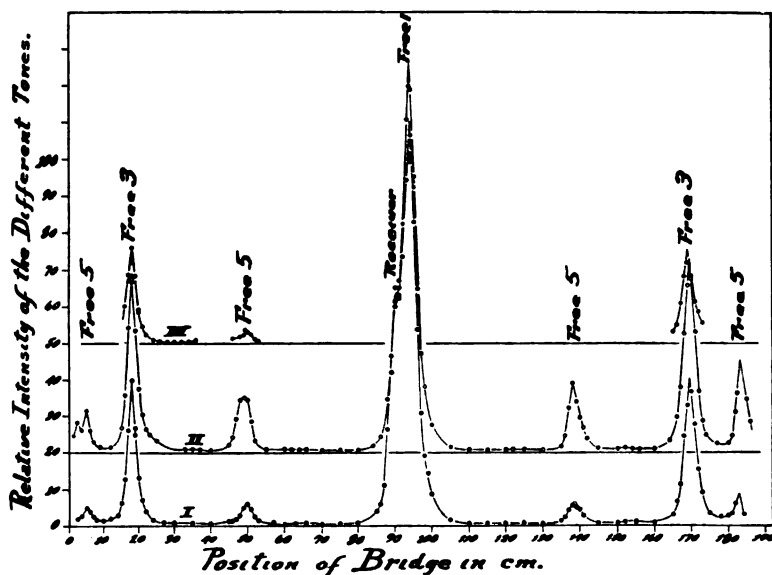


Fig. 8.

only ones present? Curve II. helps us to answer the question. Here the oscillator was longer and Table I. shows that the length for resonance is 99.9 cm. while the Lecher length was 191 cm. giving 2.08 for the ratio. This makes us believe that if a reading had been taken at 55 cm. in Curve I. it would have been at a maximum due to the oscillator fifth. Indeed the maximum at 138 cm. may easily be said to consist of two, one at

138.0, the other at 140 cm. Adding $5/3 \times 2$ to the oscillator length gives 98.7 making the ratio 2.06. The mean of these is 2.08 as before. Curve III. was partly taken when the oscillator was too short, 92.6 cm.

RESONANCE BETWEEN ALL THREE CIRCUITS.

One has only to glance at the curves of Figs. 2, 5, 7 and 8 to convince himself that as the receiver peak falls in more closely with the free first the ratio of maximum to minimum rapidly increases. Accordingly we sought to have exact resonance between the Lecher and receiver circuits and obtained it in Curve I. (black circles (•)) of Fig. 9. The only

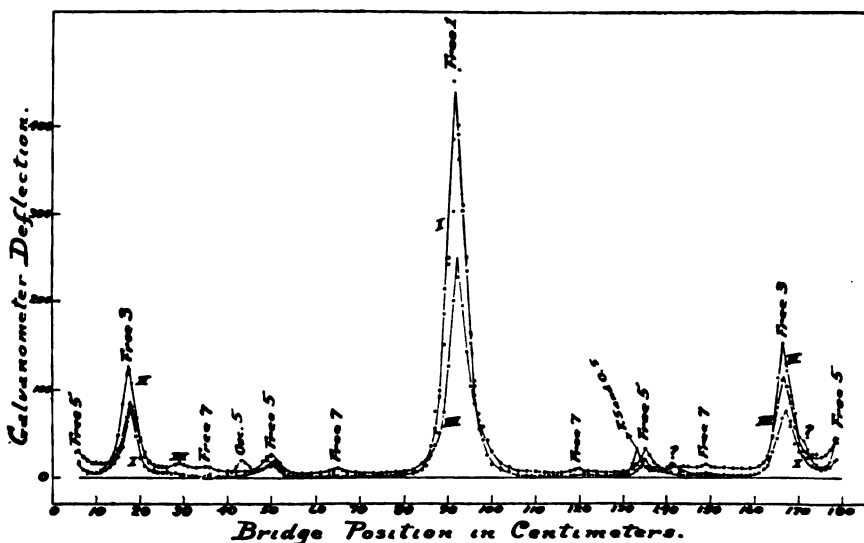


Fig. 9.

maxima present are the free first, third and fifth. But it is plain upon inspection that the free first is far too high in comparison to the other two harmonics for Abraham's theory to apply. Indeed we should not expect it to apply. For the first overtone of the receiver swings in with the free first to make it far too strong. We see no reason why when the receiver fundamental swings in with the free first its other tones should not resound to the corresponding free tones and thus render Abraham's theory applicable—on the assumption of course of proportionate damping for the different tones. We propose to try this soon.

If now the ratio 2.08 is really due to the oscillator and not to the receiver, then should the oscillator be lengthened somewhat the double peak, one due to the oscillator fifth, the other to the free fifth, should show in the 50 cm. region. Curve II. (crosses (X)) shows the double

peak. The oscillator fifth occurs at 43, the free fifth at 50. Now $7 \times 5/7 = 5$, the amount the oscillator was actually lengthened. This proves conclusively, we think, that the ratio 2.08¹ above is legitimate and in no sense due to the receiver influence; for the receiver, being in tune with the Lecher circuit, has no chance to act. Moreover we have obtained the ratio 2.08 under such a variety of conditions that the reader will scarcely be disposed to question it. It behooves us to ask ourselves then why should the oscillator have to be four per cent. longer than half the Lecher circuit length in order to cause it to respond at its best? We are aware that because of the spark-gap the damping in the oscillator circuit is greater than in the Lecher circuit, and that so far in our problem we have assumed the damping the same in the two circuits. Moreover, it is well known that the effective wave-length of an oscillation is increased by increased damping, and that accordingly the Lecher half circuit ought to be longer than the oscillator circuit to respond to it, whereas our experiments give exactly the reverse. Now it is rather remarkable that this ratio 2.08 between the oscillator and Lecher circuits is the same that both Blake and Sheard and Blake and Ruppertsberg found for the ratio of the Lecher wave-length to the Lecher wire-length for a very loose coupling. If we divide the one ratio by the other we get the ratio between the oscillator "wave-length" and the Lecher system wave-length, viz., unity. This is the very thing we ought to expect. And yet to get the oscillator "wave-length" we have multiplied its own length by 4, whereas for the Lecher system we multiplied its half-length by 4.16. Why shouldn't the oscillator circuit show this "end-correction" of 4 per cent. as well as the Lecher circuit? This all points, it seems to us, to the conclusion that this end-correction is intimately bound up with the question of the relative distribution of the three circuits concerned. And yet we have satisfied ourselves by experimentation that loosening either coupling beyond 6 cm. has no effect upon this ratio of 2.08 between the oscillator and Lecher circuits. There is one consideration that hasn't yet been taken account of, viz., What effect upon the effective oscillator length does the inpour of energy from the induction coil have? Perhaps an entirely different method of experimentation would throw some light upon this knotty question. This work shows, furthermore, that for two parallel wires without end-capacities the ratio 2.08 obtained by us and by Blake and Ruppertsberg is the limiting ratio for wave-length to wire-length of the Lecher system. This settles the question raised by us in our

¹ In Table II. for all the curves except II. and III. of Fig. 5 this ratio is 2.08. The mean oscillator length is shown in the table as 74.8. This is unquestionably too high for when the actual length was 73.5 cm. a sharp resonance was obtained for the free third at 11.5 cm. Using this length, the ratio comes out 2.08.

first paper, whether the Poincaré-Abraham ratio of 2.00 could ever be reached. We think now that it cannot, the reason being that there is an "end-correction" in the electrical case just as there is in the acoustical case, the amount of the correction being 4 per cent. Doubtless for wires of different diameters the correction varies somewhat, possibly shading off to zero for very fine wires which are *very* long compared to their distance apart.

Having found the presence of receiver influence to a greater or less degree in almost all of our curves, and having shown that the ratio of maximum to minimum is very much increased when the receiver is brought into exact tune with the other two circuits it became interesting to plot the ratio of maximum to minimum against the Lecher wire-length holding the receiver-length constant but keeping the other two circuits in tune with each other. The curve obtained is given in Fig. 10, the data for it being obtained from the other curves of this paper. As already remarked the maximum ratio was 325 for the first overtone of the receiver in tune with the other two circuits. It seems fair to expect that this ratio could be increased two or three fold by bringing the fundamental of the receiver into tune with these circuits. We expect to try this at the first opportunity.

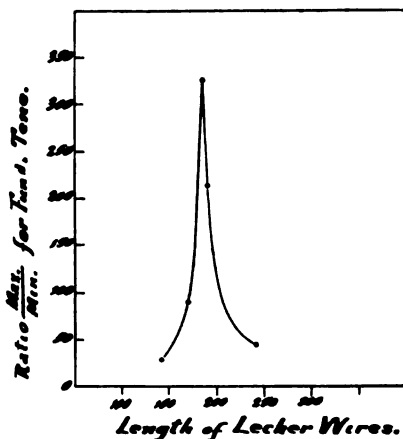


Fig. 10.

In spite of the large effect played by the receiver in our experiments as revealed especially in Fig. 10, the reader will not question, we believe, that the receiver does in no way affect the *type* of the wave system on the Lecher wires. On the other hand it surely *can* affect the relative intensity of the different harmonics, according as the receiver fundamental or some of its overtones falls in or out of phase with them. It became desirable accordingly to secure a resonance curve where the receiver might be said to be "non-selective." A first approximation to such a receiver was made by substituting for the short thermal junction wires two long ones, each 5.5 inches in length, care being taken, however, to leave the total length unchanged. That the new receiver was essentially "non-selective" was proven in the following way. 58 cm. was temporarily added to each receiver wire. The receiver peak if present ought to show itself at 110 cm. But there was apparently no trace of it, although readings

were taken by 2 cm. steps from 90 up to 130 cm. The resultant curve for this "non-selective" receiver is shown as Curve III., Fig. 9 (white circles ($\circ$)). Two things are evident by comparing Curves I. and III. of this figure. First, changing from a selective to a non-selective receiver does not affect the ratio 2.08 for the oscillator "wave-length" divided by the Lecher wire-length. Second, the free first is decreased to a normal value while the higher tones are slightly increased.

One is tempted, naturally, to try the simple theory on this Curve III. But a close inspection of this curve or indeed of any of the curves of this paper makes it clear that the agreement between theory and experiment is not good. Fig. 11 makes this plain. Calculating from the internodal

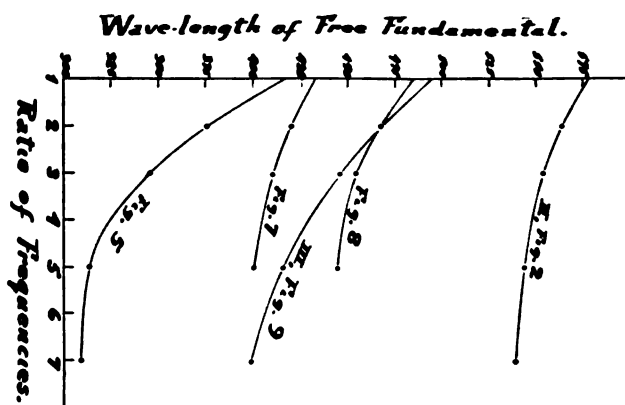


Fig. 11.

spaces for the different tones the wave-length of the free fundamental, using the resonance curves wherever possible as showing the least displacement of the various maxima, we obtain Fig. 11 and it is plain that in no case are the various tones strictly harmonic. We are in no position at present to state the reason for this. However, one or two remarks are in order, perhaps. Blake and Ruppertsberg showed (see their Fig. 6) that too close a coupling caused an inharmonic relationship amongst the different frequencies, but unlike the present case the higher the tone the greater the fundamental wave-length. It could easily be that too loose a coupling would reverse this. At any rate our experiments, all taken with a very loose coupling, viz., 6 cm. at each end of the Lecher wires, demonstrate clearly enough that the higher the tone the lower the wave-length of the free fundamental. It is to be noted, however, that more than one cause is present, for the curves of Fig. 11 do not all slope equally fast, nor does the slope change uniformly with increasing Lecher wire-length.

ARE THE CURRENTS IN A LECHER SYSTEM LONGITUDINAL
OR TRANSVERSE?

In an important paper Lecher¹ showed twenty-one years ago that if a double bridge be placed on a system of parallel wires and the wires be then cut between the two parts of the bridge, the electric potential at the ends was very slightly diminished if at all. For his potential detector Lecher used a nitrogen discharge tube, which, of course, in the nature of things is not a satisfactory quantitative instrument.

It was a comparatively easy matter for us to repeat Lecher's experiment and so while trying to clarify our notions we did so early in the work. Our apparatus was accordingly that illustrated in Fig. 7 of the former paper except that the coupling was kept constant at -2 cm. at each end of the Lecher wires throughout the work reported below. The bolometer was used as detector and a check receiver was employed.

First, we took readings both for a single bridge and for a double bridge placed in the middle of the Lecher system. The readings were the same, as we expected. Then, we soldered the double bridge to the Lecher wires and cut the Lecher wires between the parts of the double bridge. Again the readings were the same before and after cutting. We then replaced the Lecher wires by a new set consisting of two wires so arranged that the middle portion of each for a distance of 5 cm. constituted one half of the double bridge. In this way the Lecher system was divided into two equal parts which could be readily separated from each other. When the two parts of the double bridge were separated from each other by a piece of insulating tape 0.3 mm. thick, the potential at the bolometer end of the Lecher wires was 1 per cent. less than when touching. When a short piece of copper wire was soldered across each end of the double bridge parallel to the Lecher wire length, the potential was just the same as when the two parts of the bridge were touching. For these data a series of readings was taken using the check receiver. The same method was used for the curve of Fig. 12.

We then became interested to see how fast the potential at the bolometer end of the Lecher wires dropped off as the two parts of the system were separated. It was this work that forced us to resort to the metal protection of the bolometer and its lead wires. The results are shown in Fig. 12. In plotting the curves the radius of the wire, 0.04 cm., was taken into account, the abscissas representing the distance between the axes of the two wires of the double bridge. Curve I. was taken in the same manner as all the other curves of our first paper, viz., without the bolometer's being protected against stray radiation. When the bolometer

¹ Lecher, *Wied. Ann.*, 41, p. 850, 1890.

was enclosed in a metal box and its lead wires in 10-inch furnace-piping we obtained Curve II. Plotting $\log_e$ (abscissas $\times 10,000$) against the ordinates we get Curve II. *a*, showing that the potential drops off as the logarithm of the distance between the two parts of the bridge. For distances beyond 2 cm. the Curve II. *a* ceases to be a straight line. Now a considerable time after this Curve II. was taken we discovered that

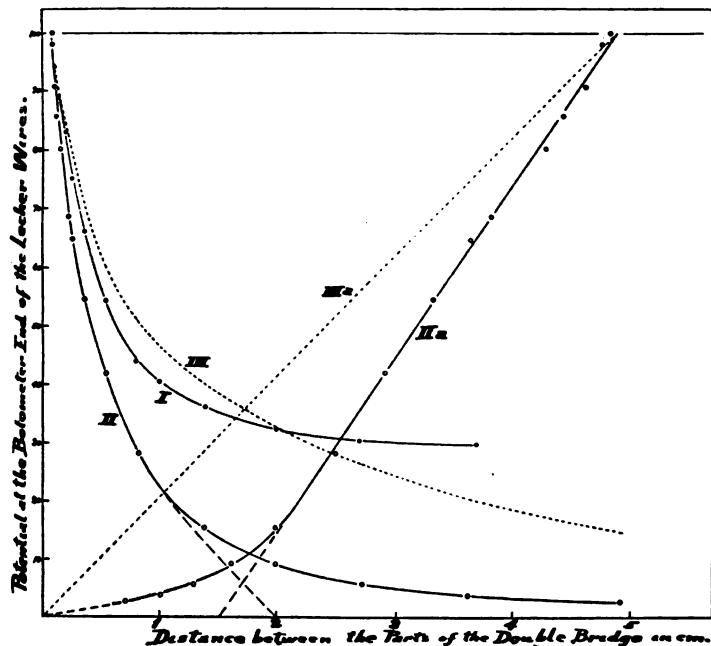


Fig. 12.

some stray radiation was striking the unprotected galvanometer lead wires and running back to upset the bolometer balance. This might have been sufficient to account for the bending of Curve II. *a* for the larger distances, and so we have continued the straight portion of II. *a* as a dotted line until it cuts the axis of abscissas. The dotted portion of II. corresponds to this. However, we do not think this could be the case, for that would mean that the potential at the bolometer end of the Lecher wires would be zero when the two parts of the bridge were 2 cm. apart whereas one would expect the potential still to be appreciable at 5 cm. just as we found it to be. Curves III. and III. *a* represent the potential in the neighborhood of a wire carrying a high frequency current, according to Abraham's theory. Considering our experimental arrangement it is perhaps too much to expect that Curves II. and III. should coincide. It is plain, however, from Curve II. that 80 per cent. of the magnetic

energy is to be found within a region of 1 cm. radius around the first part of the double bridge.

Now all this is in agreement with Lecher's experiment and with J. J. Thomson's¹ discussion of it. In other words we could say that the currents in the Lecher system are 100 per cent. transverse, for the potential at the ends of the circuit are the same whether the wires between the parts of the double bridge are cut or uncut. And yet we have had good success in treating the currents as strictly longitudinal, for we have verified Abraham's theory. Accordingly, we would like to emphasize this point. So long as the bridge is sufficiently short that the phase difference between its ends is inappreciable, it becomes *a matter of total indifference whether the currents be considered longitudinal or transverse*. For in such a case the induced current in the second half of the Lecher system neutralizes the primary current in the first half so far as the bridge itself is concerned and the two currents agree in phase. Just in proportion to the bridge length's becoming an appreciable fraction of the total length of the Lecher system so is this condition not fulfilled.

EFFECT OF SEVERAL BRIDGES.

Early in the work we had tested the effect produced by the use of several bridges, but it soon became apparent that the usual method of fixing a bridge at the maximum nearest the oscillator belonging to a given tone and then exploring with a second bridge was not the proper way to work. To be sure many experimenters have found that the internodal distance is always shortened by the employment of additional bridges and indeed we agree with that. But in the light of our experiments it is clear that all exploring with additional bridges for odd-numbered harmonics should be made by starting with a fixed bridge at the free first and working in both directions. For the even numbered harmonics, the way we proceeded is illustrated in Fig. 13.² Having located by a single bridge the free seconds at 28.0 and 122.4, we fixed a bridge at 28.0 and explored with a second bridge the 122 region. The results are shown plotted to a large scale in the figure. The maximum

¹ Recent Researches, p. 466.

² The curve is that shown as Curve III., Fig. 12, of our former paper. It differs slightly from that however, in that the check receiver readings are taken into account. The oscillator was of such a length as to swing in with the free second. The maxima marked oscillator X. were found to respond to each other and were attributed to a second oscillator vibration due to close coupling. We think now there is some doubt about this. The free third which ought to occur at 12 is apparently combined with some oscillator vibration. At any rate it occurs at 15.5 instead of 12. What we have marked Oscillator X. and Y. could perhaps be due to the bolometer, but if so this would make the bolometer circuit selective. At present it seems safer to consider it non-selective until further work shows to what extent it is selective.

occurred at 121.7 instead of 122.4, and the readings were much larger than for a single bridge. Then with the second bridge fixed at 121.7 we reset the first bridge. The maximum was found at 28.7 instead of 28, the readings being 21 per cent. higher than for the free first or 45 per cent. higher than for the average free second taken with a single bridge.

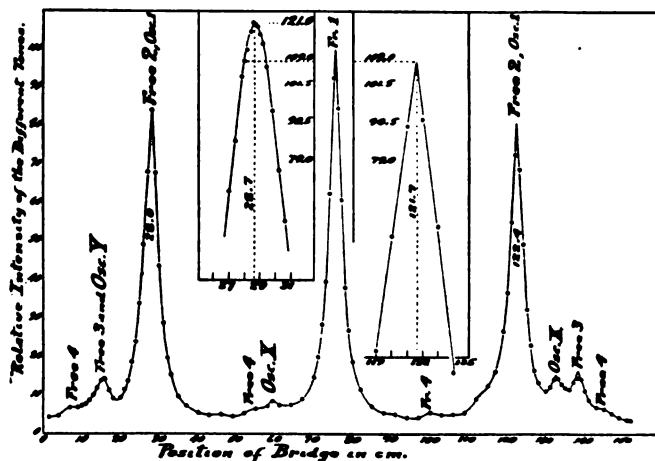


Fig. 13.

Second approximations as determined by further readjustments proved unnecessary. For instance, with a fixed internodal space of 93 cm. both bridges were moved at the same time by small steps but the readings were a maximum for the two positions above given. The internodal space for two bridges is thus 1.4 cm. shorter than for a single bridge or 1.5 per cent. less.

In a similar way, using a curve where the free thirds were prominent the internodal space was shortened from 59.0 cm. to 58.1 cm. or again 1.5 per cent. Here again it was found best to work from the middle

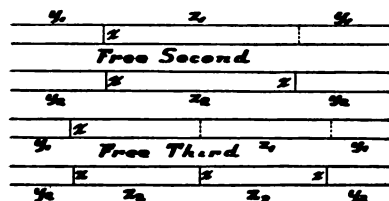


Fig. 14.

of the Lecher system outwards. That is, the middle bridge was left fixed and the other two changed. It ought to be remarked, however, that if the first bridge was left fixed and the second and third bridges changed the internodal space still remained practically 58.1 cm., although the readings were

not quite so high as when the middle bridge was left fixed.

These data are extremely important in throwing light on several things. Consider the following diagram, Fig. 14. Fixing our attention

upon the free second harmonic it would seem from reasons of symmetry that the ratio

$$\frac{z_2 + y_2}{y_2},$$

considering the currents longitudinal, should equal

$$\frac{2(z_2 + y_2) + 3x}{2y_2 + x},$$

where the currents are taken as transverse. Since

$$y_2 = \frac{z_2}{2}$$

this reduces to an identity, as it should. Moreover, using first a single bridge, then two bridges, and considering the currents transverse it would seem fair for a first approximation to say

$$\frac{2(z_1 + y_1) + x}{2y_1 + x} = \frac{2(z_2 + y_2) + 3x}{2y_2 + x},$$

which when solved gives

$$y_2 - y_1 = \frac{x}{4}.$$

Similarly for the third harmonic we would have

$$\frac{4z_1 + 2y_1 + x}{2y_1 + x} = \frac{4z_2 + 2y_2 + 5x}{2y_2 + x}$$

whence

$$y_2 - y_1 = \frac{x}{3}.$$

Now x was 5 cm., whence

$$\frac{x}{3} = 1.67 \text{ cm.}$$

and

$$\frac{x}{4} = 1.25 \text{ cm.}$$

The values for $y_2 - y_1$ by experiment were 0.45 and 0.7 cm. respectively, our experimental figures being accurate to within a mm. in each case. Now Drude¹ in an important paper has considered theoretically the effect of a bridge upon the electromagnetic conditions present in the Lecher wires. He states that a part of the energy that reaches the bridge is reflected, a part goes past it and the rest through it. In general this is surely correct. Of course, the simple theory given by Abraham takes no account of any phase changes introduced by the presence of

¹ Wied. Ann., 60, p. 11, 1897.

one or more bridges. In general it would seem that y_1 does not exactly equal $z_1/2$. And yet in all our work so far we have assumed that $y_1 = z_1/2$, strictly. Now, we have seen in our previous work and in that of Blake and Ruppertsberg that too close a coupling lengthens the internodal space of the higher tones so that they are no longer harmonic. On the other hand too loose a coupling shortens the internodal space for the higher tones and they are again not harmonic. And now we see there may be the possibility of an inharmonic relationship amongst the tones present because of a lack of symmetry of the two circuits into which the Lecher system is divided when only a single bridge is used. Of course, in the experiments reported in our earlier papers wherein we verified the Kirchhoff-Abraham generalization of the Thomson formula, it could easily have been that two of these factors so held each other in equilibrium as to make the verification possible. If this is true it means that the ideal case is to use a coupling close enough that the apparent displacement due to the coupling when a single bridge is used is just offset by the shortening of the internodal space when as many bridges are used as is the number of the harmonic under investigation.

In fact it would appear that in utter strictness Abraham's theory applies only for a bridge of zero length. As a *first approximation* it does very well for bridge-lengths up to 5 cm. but it will not do for second approximations. For the latter we are forced to accept Drude's¹ theory, rather. And in doing so we find, of course, that the shortening of the internodal space affords a satisfactory means of measuring the damping of the waves.

Using Drude's rotation, the "bridge-shortening" δb should equal $\frac{1}{2}l\zeta$, where l is the bridge-length and ζ is the ratio of the inductance of the bridge to that of one of the parallel wires per unit length. This also equals $\phi(\lambda/2)$, where λ is the wave-length of the harmonic under consideration and ϕ and ρ are two auxiliary quantities defined by the equations

$$\tan 2\pi\phi = \frac{2\pi\zeta\frac{l}{\lambda}}{1 - \gamma\zeta\frac{l}{\lambda}}$$

and

$$\rho^2 e^{2\gamma\phi} = \frac{1}{\left(1 - \gamma\zeta\frac{l}{\lambda}\right)^2 + \left(2\pi\zeta\frac{l}{\lambda}\right)^2}.$$

In these equations γ is the logarithmic decrement. Using the second

¹ Wied. Ann., 60, p. 1, 1897.

harmonic, we have

$$\phi = \frac{0.7}{93.0} = 0.00753,$$

whence $\zeta = 0.28$. For the third harmonic

$$\phi = \frac{0.45}{58.1} = 0.00774$$

and $\zeta = 0.18$. Transforming the first of these equations we have

$$\gamma = \frac{\tan 2\pi\phi - \frac{2\pi\zeta l}{\lambda}}{\tan 2\pi\phi \cdot \frac{\zeta l}{\lambda}}$$

and since

$$\phi = \frac{\zeta l}{\lambda}$$

very approximately,

$$\gamma = \frac{\tan 2\pi\phi - 2\pi\phi}{\tan 2\pi\phi \cdot \phi}.$$

Inserting the above values for ϕ for the second harmonic $\gamma = 0.100$, while for the third it is 0.103, a good agreement. The second of the equations above gives as an upper limit for γ the value 0.148. In interpreting the circuit for which we have thus measured the damping it should be remembered that we have measured the internodal space shortening for harmonics made abnormally high through resonance with the oscillator. Since in the Lecher circuit there is nothing to dissipate the electromagnetic energy, we must conclude that the logarithmic decrement for the *oscillator* was 0.102. We thus see that a very convenient method for measuring the damping of such an oscillator circuit as we have used is to let it resonate with a certain overtone of the Lecher system and measure the internodal space shortening, *but in the manner suggested above*. It is worth remarking that the "bridge shortening" is in our experiments far less than the half-bridge length, the value Drude roughly guesses at.

Drude says the "bridge shortening" can be quite appreciable without the amplitude of the wave reflected at the bridge becoming appreciable and this we agree with entirely. For the second harmonic, for instance, the ratio of the amplitudes of the reflected and incident waves comes out 0.9986. Similarly the ratio of the amplitudes of the current that passes beyond the bridge to the incident current is 0.0469. Squaring these numbers we have 0.9972 and 0.0022 which when added give 0.9994, a number which is practically equal to unity. It is well to be clear on the

interpretation of this. The current that gets past the bridge due to the incident current is very little. On the other hand, there is a large current through the bridge. This bridge-current sets up an induced current in the Lecher wires beyond the bridge. This induced current passing through the bridge in a direction opposite to the original bridge-current nullifies it almost entirely. The induced current plus the original current past the bridge make the current behind (away from the oscillator) the bridge practically equal to that before it and to the first order of approximation the bridge current is itself zero. In short, to the first order of approximation, the current in the Lecher system is *longitudinal*. Drude did not concern himself in his theory with the induced currents. We thus see that for a second and closer approximation to the facts the phase changes introduced by the presence of a bridge must be taken into account.

Further experiments along lines similar to those here presented are in progress.

RESULTS OF OTHER INVESTIGATORS.

In our earlier paper the work of some other experimenters was referred to, but for the sake of comparison it seemed better to present our own experimental work before discussing their results. We limit ourselves to the work of those investigators who touch our work pretty closely.

Cohn and Heerwagen,¹ twenty-one years ago wrote a paper in which they not only developed the generalized Thomson formula but practically verified it to a rough approximation for all tones present, viz., the first six overtones. However, they state that "the series of overtones is not an harmonic one; the internodal space is smaller than the half wavelength and in general is related to it irrationally." Their work was done with a set of Lecher wires ending in the plates of a condenser. A Geissler tube constituted their detector. By changing the distance between the plates of the end-condenser they changed the Lecher system from a closed to an open circuit and so studied the phase-changes thereby introduced. When the wires end freely they state that there is a loop of potential at their ends and that their results fit their calculations as closely as in the corresponding case in acoustics of the open organ-pipe—in other words there is an end correction. We have shown that the amount of this correction is four per cent. for the experimental conditions that we have investigated.

These authors state that their results are in fair agreement with the generalized Thomson formula if the oscillator and Lecher circuits be considered a single circuit. This is, of course, the same consideration

¹ Wied. Ann., 43, p. 343, 1891.

into which we have been forced to explain some of the minor maxima that we have obtained.

Drude¹ in his critical discussion of the mode of action of a Lecher system seems to think that the Lecher and oscillator circuits can always be considered as a single circuit. This plainly is not the case. He, too, obtained inharmonic relations amongst the various overtones.

Lecher² himself and Drude as well were aware that exploration along a system of parallel wires by means of a single bridge gave a complicated system of waves, and so Drude suggests that the relations are much simpler if two bridges are used. This no one can question. He states, however, that no matter where the first bridge is placed, the second bridge will always determine the period of the oscillator. This we are inclined to question. Considering the possibilities of interaction between the oscillator and Lecher circuits and between the Lecher and receiver circuits³ it seems best to accept the wave system whatever it is and then try to ferret it out and to investigate under what conditions it can be made simpler. It is this that we have attempted to do.

In the light of our experiments we have attempted to assign the various maxima given in the curves of Apt's⁴ paper to definite tones of the Lecher system or of the oscillator. Unquestionably most of the maxima represent the free vibrations of the Lecher circuit. The maxima of his Fig. 5 are not all easily explained. Maximum 6 is the free first, 3 and 8 are the free seconds, 1, 6 and 9 the free thirds, 5 and 7 are presumably the free fourths with two prominent maxima 2 and 4 unexplained. For his curve *a*, Fig. 6, the free thirds are at 0, 180 and 360; the free seconds at 25 and 310; the free fourths at 145 and 250 presumably, with no maxima unexplained. For all of his figures, however, even a cursory glance shows how much out of position some of the maxima are. Doubtless this displacement of the maxima is due partly at least to lack of air-gaps and water-resistance in the oscillator circuit. The relative intensities of the different harmonics are far from ideal in all cases.

One thing in Apt's paper seems entirely wrong. In discussing his Fig. 6 he says circuit *P* is in resonance with circuit *S* when the spark gap is 5 cm. from the primary plates. Correspondingly *R*₁ is in resonance with *R*₂ when the Lecher bridge is 5 cm. from the secondary plates. One ought to expect in Fig. 6 an unusually strong maximum at 5 cm. But it doesn't appear. According to our view-point the best resonance

¹ Wied. Ann., 61, p. 631, 1897.

² Wied. Ann., 41, p. 850, 1890.

³ Of course, if a Geissler tube is used as detector this last interaction would have no meaning. But such a detector at best is qualitative.

⁴ Wied. Ann., 61, p. 293, 1897.

ought to exist for all four circuits when the spark gaps and the bridge are 180 cm. from their respective capacity plates. Had the large waves from the induction coil not swamped in part the free waves of the Lecher system the maximum at 185 in his Fig. 5 would have been the strongest of all.

SUMMARY.

We have shown in this paper that both forced and free oscillations can exist simultaneously upon the Lecher system and that the overtones of the oscillator are related to each other as the odd numbers, while the free overtones of the Lecher system are related to the fundamental as the natural numbers. When the two circuits are in resonance only the odd harmonics are present to any appreciable extent.

The influence of the receiver upon the wave-system was studied, both "selective" and "non-selective" receivers being used. When the oscillator and Lecher circuits were in tune with each other and with the first overtone of the receiver the ratio of "loop-strength" to "node-strength"¹ was found to be 325 to 1.

It was found that under certain conditions the Lecher circuit and either of the other circuits behaved in part as a single circuit, and in one case all three circuits vibrated as a single circuit (Figs. 3 and 4).

It has been shown that to the first order of approximation the currents in the Lecher system can be considered either as longitudinal or transverse, the bridge-current being practically non-inductive. To the same degree of approximation Abraham's theory can be said to hold. For the second order of approximation Drude's theory of standing waves explains the facts better, though an entirely satisfactory theory must consider the induced currents in the second half of the Lecher system. By applying Drude's theory the coefficient of damping of the oscillator was measured, the method being the very convenient one of measuring the internodal space shortening.

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¹ These terms are due to Drude, l. c.

NUTATION IN PRACTICAL APPLICATIONS OF GYRO-ACTION.

BY BURT L. NEWKIRK.

IN the discussion of the phenomenon of nutation presented by Professor Franklin in the January number of this REVIEW the couple that produces the nutation is assumed to be applied suddenly. In the practical application of gyro-action it is important to take account of the fact that, owing to the non-rigidity of materials, a couple cannot be applied instantly, and to inquire what occurs when the applied couple increases from zero at a given rate. I prefer to treat the problem as follows:

Suppose a flywheel of moment of inertia I about its axle, to rotate with angular velocity ω . A coördinate system being taken as shown in the figure, and rotation about the X and Z axes being measured by the angles θ and φ respectively, we assume a couple indicated by the forces FF to increase from zero uniformly with the time. The magnitude of this couple is represented by kt . The resulting motion of the flywheel is expressed by the equations

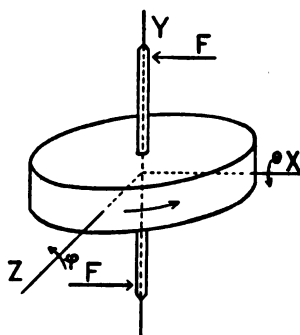


Fig. 1.

$$(1) \quad \begin{aligned} j \frac{d^2 \varphi}{dt^2} &= kt - I\omega \frac{d\theta}{dt}, \\ j \frac{d^2 \theta}{dt^2} &= I\omega \frac{d\varphi}{dt}, \end{aligned}$$

where j is the moment of inertia of the flywheel and axle about a diameter of the wheel. These equations are rigorously correct for $t = 0$ and approximately so for a short time thereafter. By having the coördinate system rotate about the X axis at a suitable rate the equations can be given a form that will represent the motion for a longer time, but the form used here is entirely satisfactory for the present purpose, since we propose to study the motion for the short time only during which a couple is being brought to bear. We assume that during this short period of time the flywheel does not move far from the position shown in the figure.

The last term in each equation expresses the effect of "gyroscopic resistance."¹

The solution of these equations is of the form

$$(2) \quad \begin{aligned} \varphi &= a \sin \lambda t + b \cos \lambda t + ct^2 + dt + e, \\ \theta &= a' \sin \lambda t + b' \cos \lambda t + c't^2 + d't + e'. \end{aligned}$$

The initial conditions are:

$$\begin{aligned} \varphi &= 0, & \theta &= 0, \\ \frac{d\varphi}{dt} &= 0, & \frac{d\theta}{dt} &= 0, \\ \frac{d^2\varphi}{dt^2} &= 0, & \frac{d^2\theta}{dt^2} &= 0. \end{aligned}$$

The remaining relations used in determining the constants are obtained by substituting expression (2) for φ and θ in (1), remembering that the resulting equations must be satisfied for all values of t so that absolute term, coefficient of t and coefficients of $\sin \lambda t$ and $\cos \lambda t$ must be separately equal to zero. We find

$$\begin{aligned} a &= -b' = e' = -\frac{kj^2}{(I\omega)^3}, \\ d &= \frac{kj}{(I\omega)^2}, \\ c' &= \frac{k}{2I\omega}, \\ \lambda &= \frac{I\omega}{j}, \end{aligned}$$

the remaining constants of integration being zero, so that the solutions are:

$$\begin{aligned} \varphi &= -\frac{kj^2}{(I\omega)^3} \sin \frac{I\omega}{j} t + \frac{kj}{(I\omega)^2} t, \\ \theta &= \frac{kj^2}{(I\omega)^3} \cos \frac{I\omega}{j} t + \frac{k}{2I\omega} t^2 - \frac{kI\omega}{j^2}. \end{aligned}$$

The period of the vibration is

$$T = 2\pi \frac{j}{I\omega}$$

and the amplitude

$$\frac{kj^2}{(I\omega)^3}.$$

¹ See *Spinning Tops, etc.*, by H. Crabtree, page 44, or *Theorie des Kreisels*, by Klein and Sommerfeld, Vol. IV., § 1.

If in place of the couple kt we have a couple C suddenly applied and constant in magnitude the solution of the equations of motion would be

$$\begin{aligned}\varphi &= -\frac{Cj}{(I\omega)^2} \cos \frac{I\omega}{j} t + \frac{Cj}{(I\omega)^2}, \\ \theta &= -\frac{Cj}{(I\omega)^2} \sin \frac{I\omega}{j} t + \frac{C}{I\omega} t.\end{aligned}$$

The period of the vibration is the same as before, but the amplitude is in general much larger since the large quantity ω which appears in the denominator is here raised to the second power only, whereas in the other case it appeared raised to the third power. To examine this point more carefully let us compare the effect of a suddenly applied couple C with the effect that would be produced if the same couple were to grow from zero to its full magnitude in τ seconds. Denoting the amplitude of the resulting vibrations by A and A' respectively we have, since $\tau k = C$,

$$\begin{aligned}A &= \frac{kj^2}{(I\omega)^2} = \frac{Cj^2}{\tau(I\omega)^2}, \quad A' = \frac{Cj}{(I\omega)^2}, \\ \frac{A}{A'} &= \frac{j}{\tau I\omega} = \frac{1}{2\tau\omega},\end{aligned}$$

if we set $j = \frac{1}{2}I$, as would be the case if the wheel were a thin disc and the mass of the axle negligible. Since the angular velocity of the fly-wheel (ω) is very large in the case of the stabilizing apparatus of the monorail car for example, and the important disturbing couples are not likely to come in the form of sharp blows the fraction A/A' will be very small, and the solution which assumes the couple to increase with the time will give a result much more in accordance with the facts than the solution based upon the ordinary theory of nutation, which assumes the couple to be instantly applied with its full force.

As a numerical example let us suppose the flywheel used to stabilize a monorail car to be a disk of nickel steel 4 feet in diameter rotating at the rate of 4,000 R.P.M. We have then the approximate value

$$\frac{A}{A'} = \frac{1}{2\tau\omega} = \frac{1}{838\tau},$$

which would be a small quantity even if τ were as small as 0.01 .

Computations of the amplitude and period of the nutational vibration for various assumed cases indicate that it will have to be given attention in the design of the monorail car, and especially in connection with the control device, but that it will probably not present any insurmountable obstacle to the development of a gyro-stabilizing system for cars.

When the impulse I of the flywheel is ample to satisfy the other requirements of the problem the nutation will be of very small amplitude and short period, so that it may be readily destroyed by damping. In the case of the monorail car the equations are less simple than those discussed above owing to the fact that the car upon which the flywheel is mounted is free to rotate about the rail, and is in unstable equilibrium. The solutions show however that the amplitude of nutation will be much less than the value corresponding to sudden application of the disturbing couple. The expression for the amplitude of the nutation when the load is applied gradually (as must be the case due to the non-rigidity of materials) contains in the denominator the quantity ω to a power one unit higher than the power of this quantity that occurs in the corresponding expression under the supposition of instantaneous application of the disturbing force.

CIRCULAR DICHROISM AND ROTATORY DISPERSION OF CERTAIN SALT SOLUTIONS.

BY L. B. OLMSTEAD.

THE object of this investigation was to study the rotation and ellipticity produced in plane polarized light by transmission through solutions containing an optically active acid radical combined with an absorbing metallic ion; and, in connection with this, the refractive indices and extinction coefficients of the same solution.

These solutions are interesting because of the discovery by Cotton¹ that certain of them show, in addition to the rotation of the plane of polarization, "circular dichroism"—that is, they absorb unequally right and left circularly polarized light, and therefore convert plane polarized light into elliptic. In all the solutions in which the latter effect is found both the ellipticity and the rotation produced are anomalous.

The work of Cotton, which was confined to copper and chromium tartrate solutions, was repeated and extended by McDowell,² who found the effect also in potassium cobalt tartrate, nickel tartrate and copper malate. Although the method used by McDowell allowed him to make observations at all parts of the spectrum, while Cotton had been limited to a few definite wave-lengths, the formulæ he used for computing ellipticities were incorrect and hence his results only qualitative.

This subject has been treated theoretically by Natanson³ for the case of mono-electronic substances—substances showing a single absorption band. He finds a very satisfactory agreement with Cotton's observations on chromium potassium tartrate but calls attention at the same time to the insufficiency of the mono-electronic theory for the majority of substances investigated.

APPARATUS.

For Measuring Ellipticities.—The Brace half-shade system was used for this purpose. The optical arrangement is shown in Fig. 1. Strong sunlight condensed by the lens L_1 on the slit S_1 of a spectral instrument was separated by the latter into a spectrum in the plane of the slit S_2 .

¹ Ann. de Chim. et de Phys., S. 7, T. 8, p. 347, 1896.

² Phys. Rev., 20, p. 163, 1905.

³ Natanson, Ladjalas, Bull. de l'Acad. des Sciences de Cracovie, Oct., 1908, Jan., 1909.

Through this slit a band of "monochromatic" light passed whose extreme wave-lengths—differing by not more than $20\mu\mu$ —depended on the width of the two slits. The lack of homogeneity of this light introduced an error in the measured ellipticity which was appreciable only at those points where there is a rapid change of direction of the ellipticity curve. Adjustments were made to reduce this error to about the same magnitude as the others inherent in the method. From S_2 the light passed successively through a condensing lens L_2 , the polarizer P , the tested solution R_2 , to the analyzing system consisting of half-shade H ,

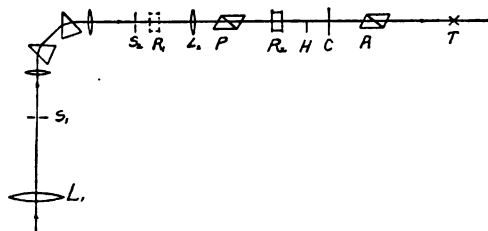


Fig. 1.

compensator C , analyzer A , and the telescope T focused on the edge of the half-shade.

The half-shade was used with its principal azimuth at 45° to that of the analyzer. The compensator was mounted on a circle graduated to measure to $.01^\circ$.

A measurement consisted in removing R_2 and placing an equal cell R_1 of the same solution before the polarizer, then adjusting the polarizer for minimum intensity of field and the compensator for a "match." This setting on plane polarized light gave the "zero" position of the compensator. The cell R_1 was then removed, R_2 inserted between polarizer and half-shade and the adjustment of polarizer and compensator carried out as before. From this rotation (φ) of the compensator the ellipticity of the light was computed—the initial setting having been made on light of zero ellipticity. The object of the auxiliary cell R_1 was to furnish the same spectral distribution of light for the zero setting as for that on the ellipticity. The rotation of the polarizer was to compensate for the rotation of the light produced by the solution. It was accomplished with sufficient accuracy by setting for minimum intensity of field.

The compensator was calibrated by Mr. Tate, of this laboratory, using Tool's¹ method. Its order was 7.56° for $550\mu\mu$. The value for other wave-lengths was obtained from Tool's dispersion curve for mica.

¹ Tool, A. Q., *PHYS. REV.*, XXXI., No. 1, p. 1, 1910.

For calculating the ellipticity Tuckerman¹ has shown that

$$E = \sin 2\theta \operatorname{tg} \pi N (\sin 2\psi - \sin 2\psi_0),$$

where θ is the angle between the major axis of the incident light and the principal azimuth of the half-shade, N is one half the order of the compensator, ψ is the angle between the principal axis of the compensator and the nicol for a match on light of the given ellipticity, ψ_0 the same for plane polarized light, and E is the ratio of the minor to the major axis of the ellipse. Since the half shade was set with its principal azimuth at 45° to the polarizer, $\sin 2\theta = 1$, and the formula reduces to the form

$$E = \operatorname{tg} \pi N [\sin 2(\varphi - \varphi_0) + \sin 2\varphi_0],$$

where $\varphi = -(\psi - \psi_0)$, and $\varphi_0 = \psi_0$. φ is obtained by rotating the compensator from a match on plane polarized light to a match on light of ellipticity E , and φ_0 by making complementary settings on plane polarized light, then subtracting half their difference from 45° . With the analyzing system used φ_0 was $6^\circ.5$ for all wave-lengths.

For Measuring Rotation.—The half-shade and analyzer were removed and the compensator replaced by a Lippich analyzing nicol, on the edge of which the telescope was focused. The measurement of rotation was then carried out as for ellipticity.

For Measuring Absorption Coefficients.—For this a Brace spectrophotometer was used in the customary way, the effect of reflection being eliminated by employing two different lengths of the solution. The extinction coefficient (nk) per wave-length in the medium was calculated from the formula

$$nk = \frac{n\lambda}{4\pi(l_2 - l_1)} \log \frac{J_1}{J_2},$$

$n\lambda$ being the wave-length (*in vacuo*) of the light, n the refractive index of the solution, and J_1/J_2 the ratio of the intensities of the light after passing through solutions of lengths l_1 and l_2 centimeters respectively.

ACCURACY.

The method used for measuring ellipticity is capable of greater accuracy than that reached by Cotton, but its principal advantage lies in its flexibility, in that readings may be made in any part of the spectrum. The rotation of the compensator could be read to $.01^\circ$, and the probable error of the mean of any set of readings was not larger than $.1^\circ$. The cells varied in length from .95 cm. to 4.40 cm., with ends of cover glasses selected free from double refraction. The values of the rotation as

¹ Tuckerman, L. B., Univ. of Neb. Stud., No. 2, 1909.

given are accurate to .02°; the extinction coefficients to 2 per cent.; and the refractive indices to 0.0003.

EXPERIMENTAL RESULTS.

The values of the rotation and ellipticity as given in the tables and curves were calculated from the observed values for *solutions one centimeter in length*. To bring all curves within the same plot the magnitude of their ordinates was changed as indicated on them. No chemical analyses of the compounds were made; the *names assigned being merely for convenience*, and not indicating that the chemical formulæ are known. The chemicals used throughout were c.p., and most of them obtained from Baker and Adamson or Kahlbaum. Most of the observations were made at a room temperature of 27° to 30° C.

Chromium Compounds.—Table I. gives values for selected wavelengths obtained from a chromium potassium tartrate solution made

TABLE I.
K₂Cr₂O₇ + Potassium Tartrate.

$\lambda(\mu\mu)$	Ellipt. $\times 10^3$.	Rot. (Deg.).	Ext. Coeff. ($n\lambda \times 10^6$).	Refr. Ind. (n).
430	— 4.3	— .73	—	—
470	3.4	— .45	4.7	1.3412
510	— 2.8	— .75	3.3	1.3391
550	—20.4	— .54	5.6	1.3374
570	—26.8	.36	7.6	1.3368
590	—21.7	.89	8.6	1.3361
610	— 6.0	1.30	8.7	1.3356
630	2.1	1.11	7.9	1.3349
650	5.5	.70	6.2	1.3344
690	4.8	.27	3.4	1.3336

according to Cotton. Two warm solutions, the one containing .750 gm. of potassium bichromate ($K_2Cr_2O_7$) and the other 10.00 gm. of neutral potassium tartrate ($K_2C_4H_4O_6$), were mixed and kept at a temperature of about 65° C. for several hours. The solution was then allowed to evaporate to dryness, after which the excess potassium tartrate was removed by fractional solution, fractional crystallization, and mechanical separation, leaving only green-colored crystals. This material was dried over P_2O_5 , and then dissolved in distilled water, the solution being made accurately in the ratio 1 : 30 by weight.

This solution gave ellipticity which has a maximum negative value at 575 $\mu\mu$, just to the left of the region of maximum absorption, which is in the yellow. (See curves plotted from the complete series of observa-

tions, Fig. 2.) The rotation becomes zero near the same point (at 565μ), remaining positive for the longer wave-lengths and negative for the shorter. The change in direction of ellipticity and the increasing negative rotation in the violet are probably due to the absorption band indicated in that region by the curve for n_k . The black circles and broken lines give the values obtained by Cotton, as calculated for the concentration used here. The agreement is very satisfactory.

By using 1.20 gm. of potassium bichromate instead of .75 gm., the crystalline mass was practically free from excess potassium tartrate, and the solution found to give nearly the same values.

The procedure for obtaining all the chromium compounds was much the same as that outlined above. When solutions of chromic acid anhydride (CrO_3) and neutral potassium tartrate are mixed and kept at

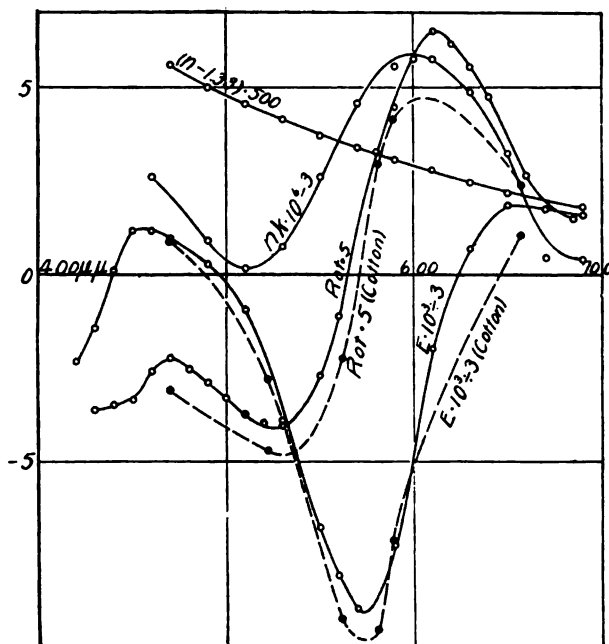


Fig. 2.

$\text{K}_2\text{Cr}_2\text{O}_7$ + potassium tartrate.

30° C. for half an hour the reaction is complete. The results in Table II. are for 1 gr. of CrO_3 to 15 gr. of potassium tartrate, the final solution being made 1 : 40. The curves (Fig. 3) are found to be very similar in form and magnitude to those obtained from the preceding solution, which was 1 : 30. In the same figure the refractive index curve of distilled water is also plotted. A comparison of this with the corresponding curve

TABLE II.
CrO₂ + Potassium Tartrate.

$\lambda(\mu\mu)$	Ellipt. $\times 10^3$.	Rot. (Deg.).	Ext. Coeff. ($nk \times 10^6$).	Refr. Ind. (n).
430	2.6	-.19	6.0	—
470	1.4	-.49	4.5	1.3402
510	- 5.0	-.72	2.9	1.3381
550	-19.1	-.64	5.1	1.3365
570	-27.0	-.09	7.1	1.3358
590	-24.6	.55	7.7	1.3351
610	-11.9	.92	7.2	1.3345
630	- 2.5	.89	6.7	1.3339
650	2.9	.59	5.3	1.3333
690	3.5	.23	2.9	1.3322

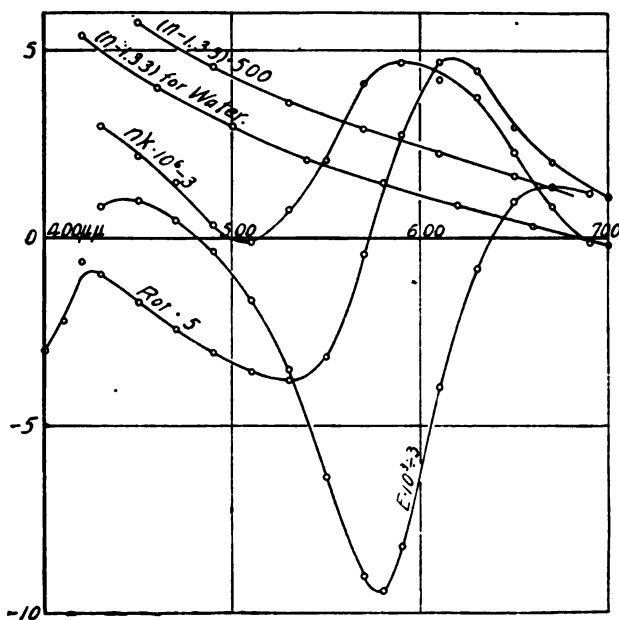


Fig. 3.

CrO₂ + potassium tartrate.

for the solution shows no trace of any anomaly, though the form of all the other curves on the plot would suggest its presence. In fact in all the solutions tested the anomaly in the refractive index curve, which theoretically should be present, is too small to be detected.

Observations made on the same stock solutions at different times indicated that the above compounds are stable.

Table III. and Fig. 4 (full lines) give the results obtained from chro-

TABLE III.
 $\text{Cr}(\text{C}_2\text{H}_3\text{O}_2)_3 + \text{Potassium Tartrate}$.

$\lambda(\mu\mu)$	Ellipt. $\times 10^3$.	Rot. (Deg.).	Ext. Coeff. ($\text{mk} \times 10^3$).	Refr. Ind. (n).
430	-1.09	.42	—	—
460	— .45	.42	5.8	1.3408
490	.36	.46	4.4	1.3391
520	1.80	.52	7.6	1.3377
550	2.87	.48	13.3	1.3366
570	3.56	.37	14.4	1.3359
600	2.12	-.07	12.5	1.3346
630	— .17	-.27	8.4	1.3333
660	— .58	-.19	5.1	1.3328
690	— .50	.09	3.2	1.3322

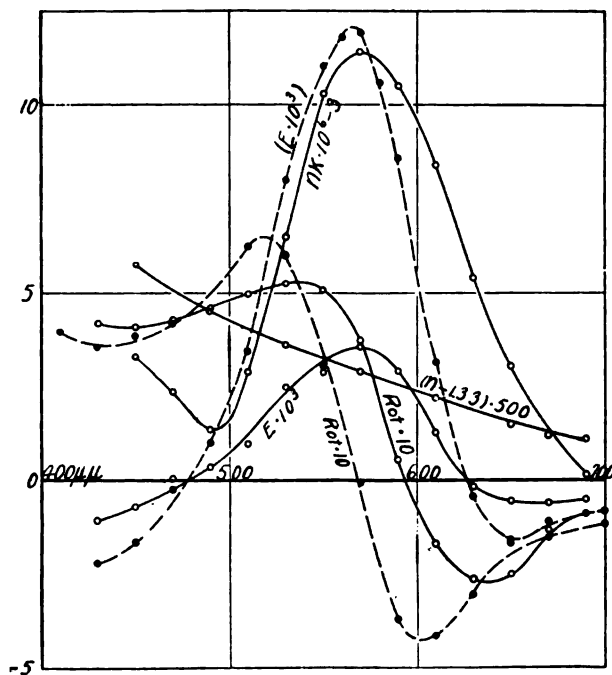


Fig. 4.

Chromium acetate + potassium tartrate.

mium acetate $\text{Cr}(\text{C}_2\text{H}_3\text{O}_2)_3$ and neutral potassium tartrate (1 : 8) dissolved in water (1 : 30). The ellipticity and rotation are opposite in sign to the other chromium solutions. This solution was also more strongly absorbing. The black circles and broken lines, representing the values of the rotation and ellipticity obtained from the same stock solution five weeks later, reveal the probable instability of the compound, although

the two sets of ellipticity curves have their maxima in the same region and pass through the zero value at the same wave-lengths. The rotation curves are however relatively displaced. The difference in the magnitude of the ellipticities is especially striking.

Table IV. and Fig. 5 give the results from a chromium potassium

TABLE IV.
CrO₃ + Potassium Lactate.

$\lambda(\mu\mu)$	Ellipt. $\times 10^3$	Rot. (Deg.).	Ext. Coeff. ($\pi \times 10^6$).	Refr. Ind. (n).
430	— .68	— .13	7.4	—
460	.47	— .10	6.7	1.3425
490	.24	— .12	5.2	1.3406
520	— 1.02	— .13	5.7	1.3390
550	— 2.71	— .07	9.1	1.3378
570	— 3.30	— .01	10.7	1.3372
600	— 2.52	.06	11.1	1.3362
630	— 1.22	.09	9.5	1.3353
660	— .60	.06	6.6	1.3345
690	— .40	.05	4.4	1.3338

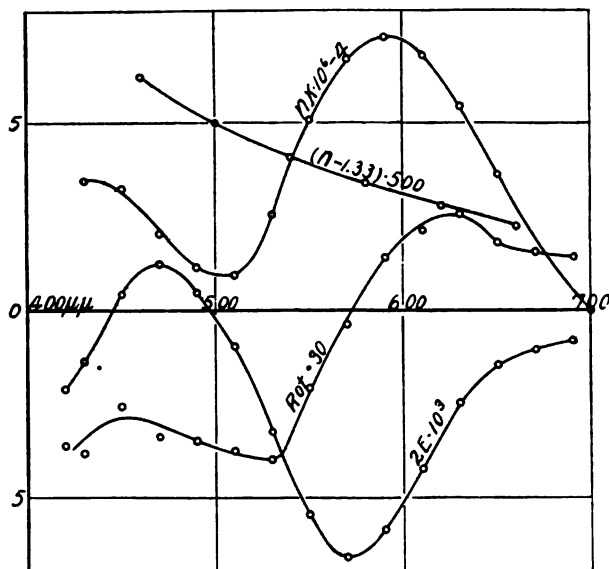


Fig. 5.

CrO₃ + potassium lactate.

lactate produced by the reaction of CrO_3 with potassium lactate (1 : 16), dilution 1 : 30. This compound was the only lactate obtained which gave circular dichroism; although trials were made with salts of copper, cobalt, manganese, nickel, iron, and uranium.

It seems likely that the above compounds are double chromium potassium tartrate salts. That the potassium (or sodium, or ammonium) is not a necessary constituent however is shown by the reaction of CrO_3 on tartaric acid. Table V. and Fig. 6 give the results from the crystal

TABLE V.
CrO₃ + Tartaric Acid.

$\lambda(\mu\mu)$	Ellipt. $\times 10^3$.	Rot. (Deg.).	Ext. Coeff. ($\text{mk} \times 10^3$).	Refr. Ind. (n).
430	— .71	.18	5.1	—
460	— .45	.17	4.3	1.3422
490	— .51	.14	3.8	1.3405
520	.78	.17	7.1	1.3392
550	.21	.20	10.8	1.3381
570	1.52	.17	10.9	1.3373
590	2.22	.10	9.3	1.3367
620	1.79	.06	6.0	1.3358
660	.99	.02	2.7	1.3347
700	.04	.01	1.4	1.3336

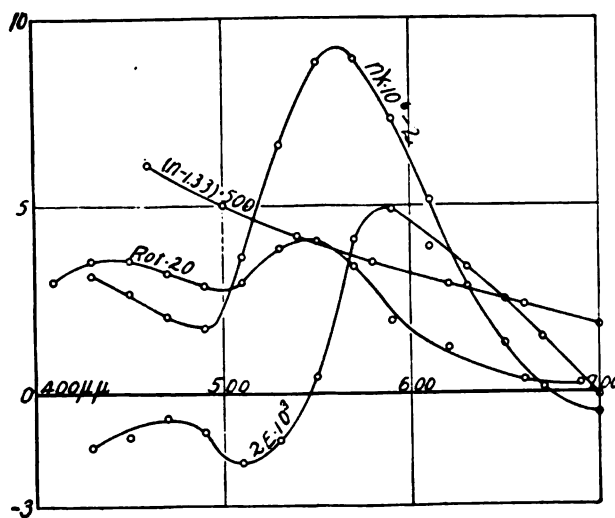


Fig. 6.

CrO_3 + tartaric acid.

formed from this reaction—1 gm. of CrO_3 on 15 gm. of tartaric acid (1 : 15), dissolved in distilled water in the ratio 1 : 30. The rotation is anomalous in form, but positive in value for all wave-lengths.

Table VI. and Fig. 7 give the values obtained from a chromium malate compound, made by the reaction of CrO_3 on malic acid (1 : 15), dilution

TABLE VI.
CrO₃ + Malic Acid.

$\lambda(\mu\mu)$	Ellipt. $\times 10^3$.	Rot. (Deg.).	Ext. Coeff. ($nk \times 10^3$).	Refr. Ind. (n).
430	.34	—	4.9	—
460	.21	-.04	2.7	1.3418
490	.09	-.04	2.7	1.3401
520	-.24	-.06	5.8	1.3388
550	-1.06	-.03	10.0	1.3377
570	-1.13	.02	10.5	1.3370
590	-.68	.04	9.4	1.3364
620	-.06	.01	5.7	1.3354
660	-.08	.01	2.8	1.3343
700	-.03	.01	1.3	1.3333

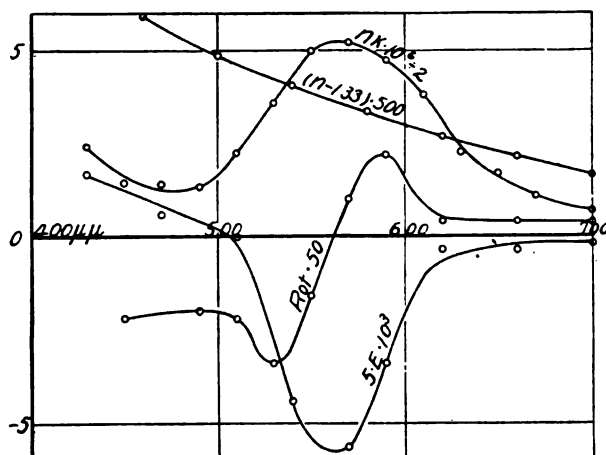


Fig. 7.

CrO₃ + malic acid.

1 : 30. The magnitude of both the rotation and the ellipticity is relatively small.

Solutions giving circular dichroism were made, using potassium chromate (K₂CrO₄), chrome alum, and chromium nitrate. Instead of potassium tartrate one may use sodium or ammonium tartrate, or Rochelle salt, or cream of tartar. The results from these were so similar to the ones given above that they are of no special interest. Reaction of CrO₃ with several other organic compounds was obtained, but none of them gave circular dichroism.

A glance at the results from the various chromium compounds shows that all the extinction coefficient curves are similar in form; the rotation is anomalous—in every case but one reverses sign in passing through

the absorption band; and the maximum ellipticity is shifted slightly from the position of maximum extinction.

Copper Compounds.—Cotton made a copper tartrate solution by dissolving 1 gm. of copper tartrate in 100 c.c. of water containing 1 gm. of KOH. The difficulty with this solution is that CuO is slowly precipitated. A similar solution was made here by using 1.50 gm. of KOH, 2.00 gm. of copper chloride, and 10.00 gm. of neutral potassium tartrate; which has the advantage in that no CuO is formed. Table VII. and Fig. 8 give the results obtained from this solution. All magnitudes are

TABLE VII.
 $\text{CuCl}_2 + \text{K}_2\text{C}_4\text{H}_4\text{O}_6 + \text{KOH}.$

$\lambda(\mu\mu)$	Ellipt. $\times 10^3$.	Rot. (Deg.).	Ext. Coeff. ($nk \times 10^6$).	Refr. Ind. (n).
430	.0	.42	.4	1.3442
470	.1	.43	.3	1.3416
510	.2	.44	.7	1.3396
550	1.6	.49	2.2	1.3380
590	4.5	.55	5.1	1.3367
620	7.9	.39	8.1	1.3358
640	8.8	.35	9.4	1.3352
660	9.7	.35	10.9	1.3348
680	9.0	.38	11.7	1.3344

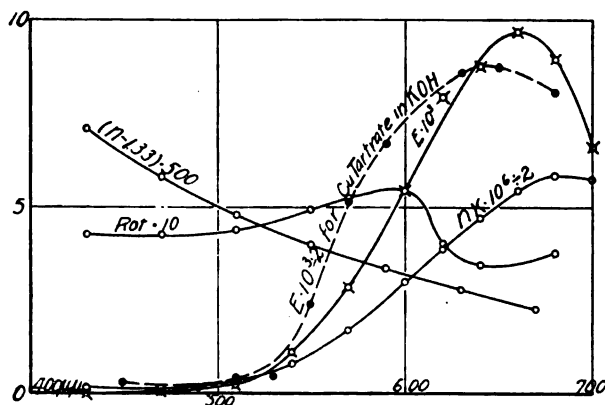


Fig. 8.
 $\text{CuCl}_2 + \text{K}_2\text{C}_4\text{H}_4\text{O}_6 + \text{KOH}.$

positive. The absorption is maximum at $680\mu\mu$, and the ellipticity at $660\mu\mu$, both decreasing toward the violet. The black circles and broken line curve is plotted from values of the ellipticity obtained from a copper tartrate solution made according to Cotton's method. The similarity is evident.

Table VIII. and Fig. 9 give the results obtained from a copper malate solution made by saturating a solution of malic acid at 20° with freshly precipitated washed copper hydroxid, evaporating the solution, and dissolving the dry material in water (1 : 30). This solution shows a strong

TABLE VIII.

Copper Malate.

$\lambda(\mu\mu)$	Ellipt $\times 10^4$.	Rot. (Deg.).	Ext. Coeff. ($\pi k \times 10^4$).	Refr. Ind. (n).
430	—	-.20	.3	1.3438
470	.12	-.17	.1	1.3415
510	.12	-.18	.3	1.3398
550	.09	-.19	1.4	1.3382
590	-.28	-.21	4.2	1.3368
620	-.82	-.25	7.8	1.3360
640	-2.03	-.28	11.2	1.3355
660	-3.00	-.27	14.8	—
680	-1.24	—	15.9	—

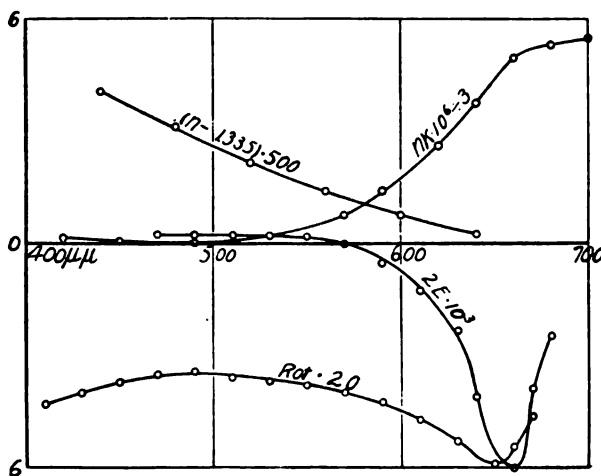


Fig. 9.

Copper malate.

absorption band in the red. The ellipticity changes through zero from a small positive magnitude in the blue and green to a sharp negative maximum in the red, at 660 $\mu\mu$. The rotation which is slightly irregular in form is negative for all wave-lengths.

Cobalt Compounds.—A dichroic potassium ammonium cobalt tartrate compound was made according to McDowell's method. A solution of cobalt chloride and neutral potassium tartrate (1 : 4) was boiled for 15 minutes and then evaporated to dryness. The red crystals were sepa-

rated out by fractional solution, and, after recrystallizing and drying, were dissolved in water (1 : 9) which contained a small amount of ammonia. This gave a wine-colored solution with an absorption band in the green. Table IX. and Fig. 10 give the results from this solution.

TABLE IX.

Cobalt Tartrate in Ammonia.

$\lambda(\mu\mu)$	Ellipt. $\times 10^3$	Rot. (Deg.).	Ext. Coeff. ($\pi k \times 10^3$).	Refr. Ind. (n).
430	1.18	—	4.3	—
460	2.10	.65	5.2	1.3524
490	4.45	.59	6.6	1.3505
520	7.15	.48	8.4	1.3490
550	8.18	.13	9.4	1.3477
580	2.48	-.42	7.4	1.3467
610	-2.21	.05	4.4	1.3458
640	-.61	.26	2.9	1.3449
670	2.83	.25	2.4	1.3441
700	3.95	.07	2.4	1.3434

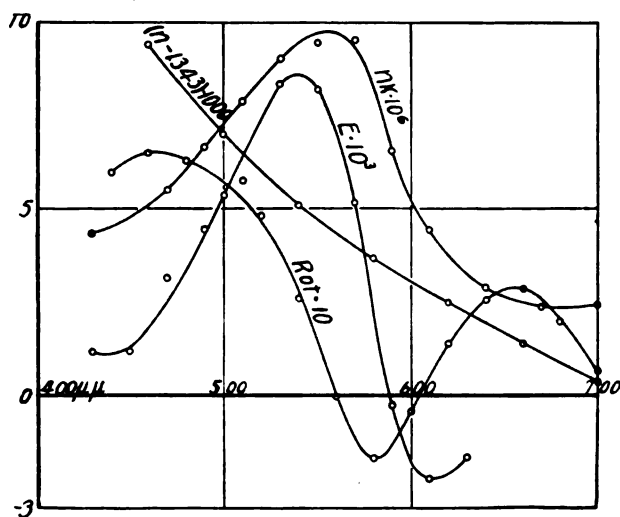


Fig. 10.

Cobalt tartrate in ammonia.

The ellipticity curve is similar in form to McDowell's, but the rotation is very different—probably owing to the fact that in the present case most of the excess potassium tartrate was removed.

Ellipticity and rotation of a different type were obtained by boiling a solution containing cobalt ammonium sulphate and ammonium tartrate (1 : 3) and dissolving in water (in the ratio 1 : 9) the red crystals so

obtained. Table X. and Fig. 11 show the results. The ellipticity which is very small reaches a maximum negative value at $500\mu\mu$ —being slightly displaced as in most of the cases, from the position of maximum absorption. The rotation is positive, and only slightly irregular in slope. Tests

TABLE X.
 $\text{Co}(\text{NH}_4)_2(\text{SO}_4)_2 + (\text{NH}_4)_2\text{C}_4\text{H}_4\text{O}_6$.

$\lambda(\mu\mu)$	Ellipt. $\times 10^3$	Rot. (Deg.)	Ext. Coeff. ($nk \times 10^6$)	Refr. Ind. (n)
430	-.04	.45	.9	1.3553
460	-.13	.37	2.5	1.3530
490	-.64	.37	4.4	1.3514
520	-.52	.35	5.7	1.3499
550	-.20	.31	4.5	1.3486
580	.12	.27	2.1	1.3476
610	.16	.24	1.2	1.3466
640	.16	.21	.9	1.3457
670	.14	.18	.9	1.3449
700	.10	.16	1.1	1.3441

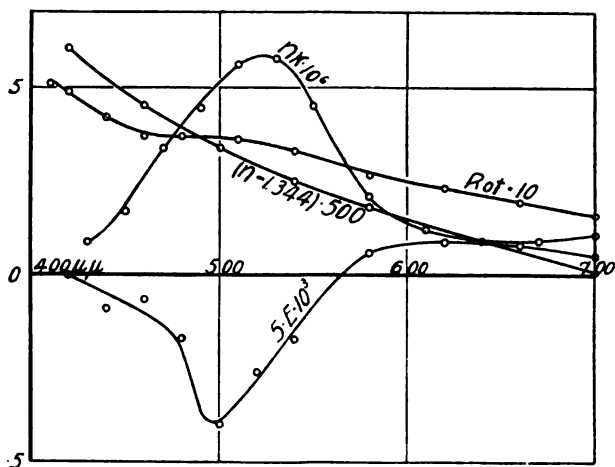


Fig. 11.

Cobalt ammonium sulfate + ammonium tartrate.

made later on the stock solutions showed that the ellipticity of both of these compounds decreased with time.

Manganese Compounds.—A saturated water solution (about 1 : 400) of manganese tartrate at about 40° was made, and to it was added a small quantity of a dilute solution of ammonia. The color changed from a very faint flesh color to red. An absorption band (see Fig. 12) undoubtedly exists in the extreme violet and another slight one in the

TABLE XI.

Manganese Tartrate in Ammonia.

$\lambda(\mu\mu)$	Ellipt. $\times 10^6$	Rot. (Deg.).	Ext. Coeff. ($\pi k \times 10^6$).	Refr. Ind. (n).
430	— .80	.18	—	1.3404
460	1.10	.19	4.9	1.3385
490	2.00	.13	4.9	1.3368
520	2.28	.07	4.9	1.3355
550	2.35	.01	4.8	1.3343
580	1.87	— .04	4.6	1.3332
600	1.13	— .06	4.1	1.3326
630	.23	— .05	3.8	1.3318
660	— .35	— .03	3.7	1.3310
700	— .38	.00	3.6	1.3300

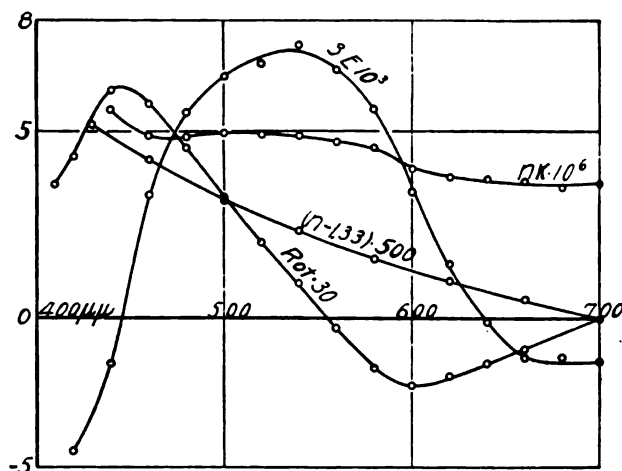


Fig. 12.

Manganese tartrate in ammonia.

green. The ellipticity changes rapidly from a negative value at $420\mu\mu$, through zero to a maximum positive value at $540\mu\mu$, decreasing again to a small negative value in the red. The effect is large for such low concentration. The rotation shows maximum positive and negative values respectively below and above $550\mu\mu$.

A similar solution was made using potassium hydrate instead of ammonia, but a red precipitate formed before observations could be made.

Attempts to obtain the nickel tartrate solution, reported by McDowell as showing circular dichroism, were unsuccessful.

No ellipticities were obtained from compounds containing nickel, iron, or uranium, although many solutions were tried.

To sum up, the colored compounds successfully used were chromium, copper, cobalt, and manganese; and the organic were tartrates, malates, and lactates, these being closely related optically active compounds.

In conclusion, the writer wishes to thank Professors Skinner and Tuckerman for the aid they have given him in this work.

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THE DEFINITION OF AN IDEAL GAS.

BY E. F. FARNAU.

INTRODUCTORY.

THE ideal gas has been treated with more or less completeness in all the texts on thermodynamics and in a considerable number of articles bearing directly upon the subject. The matter is quite worthy of consideration in that the concept of an ideal gas permits of simple though approximate formulation of the characteristic equation for a one-component body in a limited region of one-phase states, forms a basis upon which to build a more accurate characteristic equation for a one-component body, affords a convenient hypothetical scale of temperatures, and admits of ready determination of the relation between this scale of temperatures and the thermodynamic scale.

Among the more recent papers may be mentioned the following: Bakker,¹ criticizing Poincaré's² text, has deduced a relation existing between the laws of Boyle, Gay-Lussac and Joule; Baynes³ has indicated the consequences of these and other laws taken singly and two at a time; Carré⁴ and Pellat⁵ have discussed the laws of Gay-Lussac and their relation to the law of Joule; Webster and Rosanoff,⁶ purposely disregarding the law of Gay-Lussac, have formulated the laws of Boyle, Joule and of the zero Joule-Thomson effect, showing their interdependence; finally, Buckingham,⁷ in a monograph on the subject, has introduced the much-needed arbitrary scale of temperatures, and also the specific heat relations. None save the last-named author has endeavored to give a complete treatment of the subject.

The present paper is an attempt to discuss the matter from a somewhat different standpoint, in order to bring out more clearly the requirements for complete definition of the ideal gas.

¹ Bakker, *Zeit. phys. Chem.*, 14, 671 (1894); 17, 171, 678 (1895); 20, 461 (1896); 22, 543, (1897); *Jour. de Phys.* (3), 7, 152 (1898); 8, 214 (1899); *Jour. Chim. Phys.*, 4, 67 (1906); *Phys. Rev.*, 31, 589 (1910).

² Poincaré, *Thermodynamique*.

³ Baynes, *Zeit. phys. Chem.*, 18, 335 (1895); 21, 556 (1896); *Proc. Phys. Soc. Lond.*, 15, 361.

⁴ Carré, *Jour. de Phys.* (3), 7, 718 (1898).

⁵ Pellat, *Jour. de Phys.* (3), 8, 100 (1899).

⁶ Webster and Rosanoff, *Phys. Rev.*, 29, 304 (1909).

⁷ Buckingham, *Bull. Bur. Stands.*, 6, No. 3 (1910).

THE LAW OF BOYLE.

Concerning the statement of this law no misunderstanding has arisen. It is given briefly as follows: The volume of unit mass of any gas at a fixed temperature is inversely proportional to the pressure. It may be formulated

$$(1) \quad pv = F(\tau),$$

where p is the pressure, v the volume of unit mass of the gas, and $F(\tau)$ some function of the temperature τ of an arbitrary temperature scale. τ is some uniform increasing or decreasing function of the degree of hotness of the body.

THE LAWS OF GAY-LUSSAC.

(a) *Formulation of the Laws of Gay-Lussac.*—Here considerable differences of opinion exist. The following are current statements of the laws: The pressure being constant, the volume of a unit mass of gas varies directly as its absolute temperature (the temperature being implicitly considered to be measured on the thermodynamic scale); and, the volume of a unit mass of gas being constant, the pressure varies as the absolute, i. e., thermodynamic, temperature. It is obvious that historically at least there is no justification for such definitions. At the time of Gay-Lussac, the scale of temperatures was defined by assumption of the constancy, between the two fixed temperatures of melting ice and boiling water, of the volume-temperature derivative of some arbitrary substance—mercury, alcohol, or one of the “permanent” gases. That the thermodynamic scale of temperatures later defined by Kelvin should for a wide range agree so closely with that of the constant-pressure gas thermometer has nothing to do with the observations of Gay-Lussac.

A correct statement of Gay-Lussac's laws is: For constant pressure, the relative thermal expansion of a mass of gas between two assigned temperatures is a constant, which is independent of the constant pressure, and of the nature of the gas; and, for a constant volume of a mass of gas, the relative pressure increase between two assigned temperatures is a constant, which is independent of the constant volume, and of the nature of the gas. The two laws are formulated as follows:

$$\frac{v(p, \tau_1) - v(p, \tau_0)}{v(p, \tau_0)} = \alpha; \quad \frac{p(v, \tau_1) - p(v, \tau_0)}{p(v, \tau_0)} = \beta.$$

Gay-Lussac observed that α and β were approximately equal for the particular gases under the particular conditions of his experiments. It will be noted that equality of α and β really constitutes a third law of

Gay-Lussac, since this is altogether independent of the above statements of the two laws. However, in order not to complicate matters unnecessarily, this assumption shall in the following be implicitly made.

(b) *Definition of the Scale of Temperatures of the Constant-pressure Gas Thermometer.*—Taking τ_0 as the temperature of melting ice and $\tau_0 + 100$ as the temperature of boiling water—i. e., taking 100 degree-divisions of the τ -scale of temperature between these two fixed points—it is found (Gay-Lussac obtained the value 0.391, later corrected to 0.366) that under atmospheric pressure

$$\frac{v(p, \tau_0 + 100) - v(p, \tau_0)}{v(p, \tau_0)} = 0.366.$$

If, now, any temperature τ on this scale be considered such that $\tau - \tau_0$ is proportional to the volume increase relative to the volume at the temperature of melting ice, then

$$k(\tau - \tau_0) = \frac{v - v_0}{v_0} \text{ at constant pressure.}$$

When

$$\tau = \tau_0 + 100, \quad k = 0.00366.$$

The relation is further simplified by writing $\tau_0 = 273$; then

$$\tau = \frac{273}{v_0} v.$$

A similar definition of the temperature scale of a constant-volume gas thermometer could be stated:

$$\tau = \frac{273}{p_0} p.$$

The two equations for the τ -temperature of a body are also formulations of the two laws of Gay-Lussac, and as such will be employed hereafter in place of the more cumbersome forms first stated.

Since the quantities v_0 and p_0 are functions of p and v respectively, the two laws of Gay-Lussac can be written

$$(2a) \quad \frac{v}{\tau} = \psi_1(p),$$

$$(2b) \quad \frac{p}{\tau} = \psi_2(v).$$

THE FREE-EXPANSION EFFECT.

(a) *Zero Free-expansion Effect*.—When a mass of one of the “permanent” gases expands adiabatically into a vacuum, the temperature of the mass remains practically unchanged. This was first observed by Gay-Lussac, and again much later by Joule. Assuming the temperature to remain constant,

$$\tau(e, v_2) - \tau(e, v_1) = 0,$$

where e is the specific energy of the gas; and

$$e(v_2, \tau) - e(v_1, \tau) = 0;$$

whence it appears that both τ and e are independent of v , i. e.,

$$(3) \quad \left(\frac{\partial \tau}{\partial v} \right)_e = 0,$$

and

$$\left(\frac{\partial e}{\partial v} \right)_\tau = 0.$$

(b) *Formulation of the Free-expansion Effect*.—In order to obtain general expressions for these two derivatives, the procedure is as follows: If in the differential equation for the energy expressed as a function of v and s

$$de = -p dv + \theta ds,$$

where s is the specific entropy of the body, and $\theta(\tau)$ is the thermodynamic temperature, the variables v, τ be taken independent, the equation becomes

$$de = \left(-p + \theta \frac{\partial s}{\partial v} \right) dv + \theta \frac{\partial s}{\partial \tau} d\tau,$$

and the condition for an exact differential yields

$$-\frac{\partial p}{\partial \tau} + \frac{d\theta}{d\tau} \frac{\partial s}{\partial v} = 0;$$

whence

$$(4) \quad \left(\frac{\partial \tau}{\partial v} \right)_e = \frac{- \left(-p + \theta \frac{\partial p}{\partial \tau} \frac{d\tau}{d\theta} \right)}{c_v(v, \tau)},$$

since $\theta(\partial s / \partial \tau) = c_v(v, \tau)$, the specific heat at constant volume per degree of temperature τ . Also

$$\left(\frac{\partial e}{\partial v} \right)_\tau = \left(-p + \theta \frac{\partial p}{\partial \tau} \frac{d\tau}{d\theta} \right).$$

(c) *The Relation between the Thermodynamic Temperature and the Free-expansion Effect.*—Solving equation (4) for θ :

$$\frac{d \log \theta}{d\tau} = \frac{\frac{\partial p}{\partial \tau}}{p - c_v(v, \tau) \left(\frac{\partial \tau}{\partial v} \right)_\theta}.$$

When the quantities in the second member of this equation are experimentally determined with sufficient accuracy, as functions of the variables v, τ , the relation between the arbitrary scale of temperature and the thermodynamic scale is established.

(d) *Consequence of the Law of Zero Free-expansion Effect.*—From the above equation, writing

$$\left(\frac{\partial \tau}{\partial v} \right)_\theta = 0$$

it follows, since $c_v(v, \tau)$ is always finite, that

$$\frac{d \log \theta}{d\tau} = \frac{\partial \log p}{\partial \tau},$$

whence on integrating

$$(5) \quad \frac{p}{\theta} = f_1(v).$$

This is an equivalent expression of the law of zero free-expansion effect.

THE POROUS-PLUG EFFECT.

(a) *Zero Porous-plug Effect.*—The measurement of the free-expansion effect necessitates the use of bulky apparatus, and, since the effect is at best very small, its measurement is extremely uncertain. In order to determine with greater accuracy the temperature change in an expansion, Joule and Thomson devised the so-called porous-plug experiment. In this operation a steady state is reached during which gas, under a constant initial pressure, is forced through a porous plug, and escapes under a constant final pressure—the whole apparatus being thermally insulated.

Here the work $p v$ is added to unit mass of the gas in the initial stage of the operation, while the work $(p + \delta p)(v + \delta v)$ is developed by the gas in the concluding stage. The process being adiabatic, the change of energy is

$$\delta e = - \delta(pv),$$

that is,

$$\delta(e + pv) = 0, \quad e + pv = \text{const.}$$

This is the condition under which the experiment is made; and the derivative

$$\left(\frac{\partial \tau}{\partial p}\right)_{e+pv}$$

is called the porous-plug effect. Under different conditions of pressure and temperature this may be positive, zero, or negative. It is of interest to determine the consequences of assumption of a zero porous-plug effect, *i. e.*,

$$(6) \quad \left(\frac{\partial \tau}{\partial p}\right)_{e+pv} = 0.$$

An equivalent expression is

$$\left(\frac{\partial(e + pv)}{\partial p}\right)_\tau = 0.$$

(b) *Formulation of the Porous-plug Effect.*—From the differential equation for the specific energy, we obtain, on adding $d(pv)$ to both members,

$$d(e + pv) = vdp + \theta ds.^1$$

Taking p, τ as independent variables,

$$d(e + pv) = \left(v + \theta \frac{\partial s}{\partial p}\right) dp + \theta \frac{\partial s}{\partial \tau} d\tau;$$

and the condition for an exact differential yields

$$-\frac{\partial s}{\partial p} \frac{d\theta}{d\tau} = \frac{\partial v}{\partial \tau};$$

whence

$$(7) \quad \left(\frac{\partial \tau}{\partial p}\right)_{e+pv} = \frac{-\left(v - \theta \frac{\partial v}{\partial \tau} \frac{d\tau}{d\theta}\right)}{c_p(p, \tau)};$$

since $\theta(\partial s/\partial \tau) = c_p(p, \tau)$, which is the specific heat at constant pressure per degree of temperature τ .

(c) *Relation between the Thermodynamic Temperature and the Porous-plug Effect.*—Solving equation (7) for θ :

$$\frac{d \log \theta}{d\tau} = \frac{\frac{\partial v}{\partial \tau}}{v + c_p(p, \tau) \left(\frac{\partial \tau}{\partial p}\right)_{e+pv}}.$$

¹ This fundamental function $(e + pv) = g(p, s)$ is the χ -function of Gibbs, called by him the heat function at constant pressure. Kammerlingh-Onnes has suggested for it the name "enthalpy."

When the quantities in the second member of this equation are experimentally determined with sufficient accuracy, as functions of the variables p , τ , the relation between the arbitrary scale of temperatures and the thermodynamic scale is established.

(d) *Consequence of the Law of Zero Porous-plug Effect.*—From the above equation, writing

$$\left(\frac{\partial \tau}{\partial p}\right)_{\theta, pv} = 0,$$

it follows, since $c_p(p, \tau)^1$ is always finite, that

$$\frac{d \log \theta}{d \tau} = \frac{\partial \log v}{\partial \tau};$$

whence on integrating

$$(8) \quad \frac{v}{\theta} = f_2(p).$$

This is an equivalent expression of the law of zero porous-plug effect.

THE CHARACTERISTIC EQUATION OF AN IDEAL GAS.

It is now possible to deduce consequences of the laws of Boyle, of Gay-Lussac, of zero free-expansion effect, and of zero porous-plug effect, taken two at a time.

(a) The laws of Boyle and zero free-expansion effect lead at once to the relation $pv = r\theta$; for, eliminating p between equations (1) and (5),

$$pv = F[\tau(\theta)], \quad \frac{p}{\theta} = f_1(v),$$

and separating the variables,

$$\frac{F[\tau(\theta)]}{\theta} = f_1(v) \cdot v = r;$$

whence

$$pv = r\theta.$$

This is the so-called "characteristic equation" of an ideal gas. It is a relation between the pressure, specific volume, and thermodynamic temperature of a one-component, one-phase body, approximately valid in a limited region of vapor states of the body.

(b) The laws of Boyle and of zero porous-plug effect in similar manner lead at once to the relation $pv = r\theta$; as is seen by inspecting equations (1) and (8),

$$pv = F[\tau(\theta)], \quad \frac{v}{\theta} = f_2(p).$$

¹ Here c_p denotes the specific heat of an ideal gas.

(c) The first law of Gay-Lussac and the law of zero free-expansion effect, when combined, lead to no simple expression.

The second law of Gay-Lussac and the law of zero free-expansion effect require the temperatures of the thermodynamic scale and those of the constant-volume gas thermometer to be proportional for all volumes. See equations in section e.

(d) The first law of Gay-Lussac and the law of zero porous-plug effect require the temperatures of the thermodynamic scale and those of the constant-pressure gas thermometer to be proportional for all pressures. See equations in section e.

The second law of Gay-Lussac and the law of zero porous-plug effect when combined lead to no simple expression.

(e) Just as the laws of Boyle and the two laws of Gay-Lussac are interdependent, so are the laws of Boyle, of zero porous-plug effect, and of zero free-expansion effect; as is evident by inspection of the following equations:

$$(1) \quad pv = F(\tau), \qquad (1) \quad pv = F[\tau(\theta)],$$

$$(2a) \quad \frac{v}{\tau} = \psi_1(p), \qquad (8) \quad \frac{v}{\theta} = f_1(p),$$

$$(2b) \quad \frac{p}{\tau} = \psi_2(v), \qquad (5) \quad \frac{p}{\theta} = f_2(v).$$

SPECIFIC HEATS.

It will be shown later, that in addition to the characteristic equation, some knowledge of specific heats is necessary to define a one-component body in a given region of states.

(a) *The Difference of the Specific Heats.*—Evaluation of the general relation

$$c_p - c_v = \theta \left(\frac{\partial p}{\partial \theta} \right)_v \left(\frac{\partial v}{\partial \theta} \right)_p,$$

by means of the characteristic equation $pv = r\theta$, leads at once to the result

$$c_p - c_v = r.$$

(b) *Regnault's Law and its Consequences.*—It was observed by Regnault that, for the "permanent" gases, the specific heat at constant pressure is practically constant, *i. e.*, independent of either pressure or temperature,

$$c_p(p, \theta) = \text{const.}$$

Thence follows that

$$\frac{\partial c_p}{\partial p} = 0 \quad \text{and} \quad \frac{\partial c_p}{\partial \theta} = 0.$$

For want of better names, these formulations will be called statements of Regnault's first and second laws, respectively.

The first law permits partial determination of the characteristic equation; for, since in general

$$\frac{\partial c_p}{\partial p} = -\theta \frac{\partial^2 v}{\partial \theta^2},$$

it follows on integration that

$$(9a) \quad \frac{\partial v}{\partial \theta} = \varphi_1(p),$$

which is the differential form of the consequence of the law of zero porous-plug effect.

In an analogous manner, if one expresses a law of constancy of the specific heat at constant volume, it follows that

$$\frac{\partial c_v}{\partial v} = 0 \quad \text{and} \quad \frac{\partial c_v}{\partial \theta} = 0,$$

and from the first equation

$$\frac{\partial c_v}{\partial v} = 0 = \theta \frac{\partial^2 p}{\partial \theta^2};$$

whence

$$(9b) \quad \frac{\partial p}{\partial \theta} = \varphi_2(v),$$

which is the differential form of the consequence of the law of zero free-expansion effect.

Moreover the law of Boyle and the consequences of the two laws $\partial c_p / \partial p = 0$ and $\partial c_v / \partial v = 0$ show the same interdependence as do the law of Boyle and the two laws of Gay-Lussac; or the law of Boyle, the law of zero free-expansion effect, and the law of zero porous-plug effect—as is evident on comparison of the groups of equations: (1), (2a), (2b); (1), (8), (5); (1), (9a), (9b).

On the other hand the equation

$$\frac{\partial c_p}{\partial \theta} = 0$$

is in no manner connected with the characteristic equation. The same statement applies to the equation

$$\frac{\partial c_p}{\partial \theta} = 0.$$

It is obvious, therefore, that either of these equations affords knowledge of the thermodynamic properties of a body, not to be obtained from the characteristic equation. Yet, since $(c_p - c_v)$ depends solely upon the form of the characteristic equation, the temperature derivative of either can be found from that of the other.

FUNDAMENTAL FUNCTIONS OF A ONE-COMPONENT, ONE-PHASE BODY.

Gibbs¹ has shown that for any one-component body in any region of one-phase states, there exist three independent relations between the six variables p, θ, h, V, S, M (the pressure, temperature, potential, volume, entropy and mass of the body). Any definition from which three such independent relations can be deduced is a complete definition of the body. Further, he has shown that a relation between $e, v, s; f, v, \theta; g, p, s$; or h, p, θ constitutes such a definition (where f is the specific free-energy of the body).

(a) *Sufficient Relations for the Formulation of the Fundamental Functions.*—To illustrate the above statement, consider, for example, the specific free-energy of the body. Then

$$df(v, \theta) = \frac{\partial f}{\partial v} dv + \frac{\partial f}{\partial \theta} d\theta = -p dv - s d\theta,$$

and

$$ds(v, \theta) = \frac{\partial s}{\partial v} dv + \frac{\partial s}{\partial \theta} d\theta = \frac{\partial p}{\partial \theta} dv + \frac{c_v}{\theta} d\theta,$$

and also

$$dc_v(v, \theta) = \frac{\partial c_v}{\partial v} dv + \frac{\partial c_v}{\partial \theta} d\theta = \theta \frac{\partial^2 p}{\partial \theta^2} dv + \frac{\partial c_v}{\partial \theta} d\theta.$$

Consequently, if the characteristic equation together with the form of $\partial c_v(v, \theta)/\partial \theta$ is known, one has sufficient data with which to obtain the specific free energy as a function of v, θ ; and hence to formulate the thermodynamic properties of the body in that region of states for which the characteristic equation and the expression for the specific heat derivative are valid.

In similar manner, the specific thermodynamic potential can be determined from the characteristic equation and the form of $\partial c_p(p, \theta)/\partial \theta$.

¹ Gibbs, Scientific Papers, I., p. 88.

(b) *The Definition of an Ideal Gas.*—From the above two examples, it is seen that the form of the characteristic equation and the form of either $\partial c_v / \partial \theta$ or $\partial c_p / \partial \theta$ are sufficient to define a one-component body thermodynamically in a region of its one-phase states. In the particular case of the ideal gas there is a common consensus of opinion that the characteristic equation $p v = r \theta$ is an essential feature of its definition, but the further necessity of the second law of Regnault—that the temperature derivative of one of the specific heats is zero—is generally neglected.

As an illustration, consider the behavior of two gases on reversible adiabatic expansion, both gases having the characteristic equation $p v = r \theta$, while the specific heat temperature derivative of the first is zero, and that of the second is constant and finite,

$$\text{I. } \frac{\partial c_v}{\partial \theta} = 0, \quad \text{II. } \frac{\partial c_v}{\partial \theta} = b_2.$$

Then for both:

$$\begin{aligned} ds &= \frac{\partial s}{\partial v} dv + \frac{\partial s}{\partial \theta} d\theta \\ &= \frac{\partial p}{\partial \theta} dv + \frac{c_v}{\theta} d\theta, \end{aligned}$$

and for each, since

$$\text{I. } c_v = a_1, \quad \text{II. } c_v = a_2 + b_2 \theta;$$

therefore

$$ds = \frac{r}{v} dv + \frac{a_1}{\theta} d\theta, \quad ds = \frac{r}{v} dv + \frac{a_2 + b_2 \theta}{\theta} d\theta,$$

and

$$\left(\frac{\partial v}{\partial \theta} \right)_s = -\frac{v a_1}{r \theta}, \quad \left(\frac{\partial v}{\partial \theta} \right)_s = -\frac{v}{r \theta} (a_2 + b_2 \theta).$$

These differing results indicate the insufficiency of the equation $p v = r \theta$ as a definition of an ideal gas.

SUMMARY.

The preceding is an attempt at logical development of the concept of an ideal gas. After the many discussions of the subject in the past, it is needless to state that no original conclusions are reached; it is hoped merely that the method of presentation has some advantages. It is a pleasure to acknowledge my indebtedness to Professor Trevor for his criticism and advice in the preparation of this paper.

CORNELL UNIVERSITY,
January, 1912.

A RELATION BETWEEN THE MAGNETIC AND ELASTIC PROPERTIES OF A SERIES OF UNHARDENED IRON-CARBON ALLOYS.

BY C. W. WAGGONER.

THIS paper is a continuation of a study of some of the physical properties of a series of annealed iron-carbon alloys; the magnetic hysteresis and coefficient of linear expansion having been presented in a previous paper.¹ For convenience the chemical analysis and heat treatment have been repeated here in Table I. below.

TABLE I.
Chemical Analysis.

Mark.	C	P	Si	Mn	S
P.I.	.058	Trace	.008	.071	—
A1	.60	.013	.15	.14	.012
A2	.74	.012	.16	.14	.013
A3	.89	.010	.19	.155	.013
A4	.98	.012	.16	.15	.013
A5	1.18	.012	.14	.14	.013
A55	1.26	.012	.16	.17	.014
A6	1.37	.011	.19	.16	.012

Heat Treatment.—The steels were all placed in a platinum resistance furnace and heated to 1000° C. After keeping them at that temperature for two hours, they were allowed to cool slowly in the furnace.

THE MAGNETIC HYSTERESIS AT 20° C.

In the previous investigation it was found that the hysteresis loss, at ordinary temperatures, increases with the carbon content up to about 1.1 per cent. of carbon and then decreases with increasing carbon. The limited series of steels, and the fact that the results did not indicate clearly the maximum of the curve of hysteresis loss and the percentage of carbon, led the writer to plot some data given by Benedicks² for a series of annealed steels. Benedicks' data when plotted showed a decided

¹ Waggoner, *Phys. Rev.*, Vol. XXVIII., p. 393, 1909.

² Benedicks, *Thèse Pour Le Doctorat*, Uppsala, 1904.

maximum and the writer felt justified in drawing his curve in a like manner. These curves are shown in Fig. 1 as taken from a previous paper. The fact that the maximum of the hysteresis-carbon curve should lie so near the eutectic point of the iron-carbon series, was pointed out at the time, but the writer did not feel that the data were sufficient to justify any speculation as to the significance of this fact. Further study has, however, led to the conclusion that the shape of the hysteresis-carbon curve may be explained by a consideration of the physical properties of the microscopic constituents of the iron-carbon series.

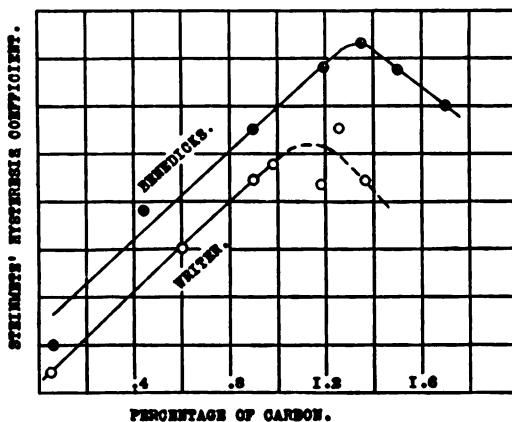


Fig. 1.

Showing the relation between the hysteresis loss and carbon content.

Sauver¹ calculated the proportions of the three constituents present in unhardened steels from an analysis of cementite and pearlite. A table showing the results of more recent investigation is however, given by Howe² and is reproduced in Fig. 2. An examination of the constituents of unhardened steels as given in Fig. 2 shows pure iron to be made up of a microscopic constituent called Ferrite.

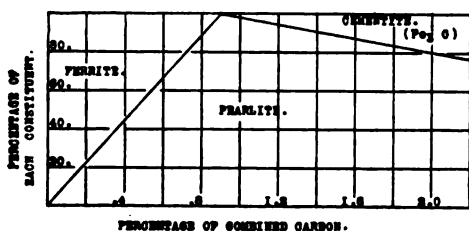


Fig. 2.

Showing the constitution of iron-carbon compounds.

This microscopic constituent is extremely brittle, harder than hardened steel, and magnetic (but probably much less so than ferrite).

¹ Sauver, Am. Inst. of Min. Eng., Sept., 1896.

² Howe, Iron, Steel and other Alloys, p. 184.

³ Loc. cit., p. 175.

In a steel containing about 1 per cent. carbon (the eutectic) ferrite and cementite form in very closely interstratified layers, representing perhaps the most intimate mixture of these two constituents possible. This combination of ferrite and cementite, because of its appearance under the microscope, has been called pearlite. From the figure it will be seen that a low carbon steel consists of a mixture of this pearlite and an excess of pure iron or ferrite. As the eutectic percentage of carbon and iron is approached the amount of free ferrite grows less, and at 1 per cent., or near it, the alloy consists entirely of this closely-woven mixture of ferrite and cementite. Beyond the eutectic percentage the alloy consists of pearlite and an excess of cementite.

Assuming now, that the hysteresis loss in an alloy is due to the magnetic reversal of polarity of the iron molecules when placed in an alternating field, it seems probable that the increase in the hysteresis loss up to the eutectic may be due to the decrease in free ferrite, and the increase in pearlite; the opposition to the reversal of polarity increasing with the decreasing percentage of free ferrite. It seems reasonable, also, to suppose that the magnetic molecules of cementite and the molecules of ferrite, bound up so closely to form pearlite, would not possess the freedom of molecular vibration under the alternating magnetic field as would free, uncombined ferrite molecules. Following further this method of reasoning, the hysteresis loss should be a maximum at the eutectic percentage, since at this point we have ferrite and cementite forming a mixture so intimate that the separate layers may only be detected microscopically when very high magnifications are used.

That the hysteresis loss decreases, when the carbon content is in excess of the eutectic percentage, may be due to the freeing of the ferrite molecules in the solid mass of pearlite by the net work of excess cementite which surrounds the pearlite crystals and separates them from each other, making it easier for the ferrite molecules to follow the alternations of the impressed magnetic field. Of course this decrease in hysteresis loss does not continue with increasing cementite but must reach a minimum, where the advantage gained by loosing the bound molecules of ferrite is met by the decreasing number of them present in the alloy.

INTENSITY OF MAGNETIZATION.

From the reasoning given above for the hysteresis-carbon curve it is clear that the intensity of magnetization, at any given magnetic field strength high enough to produce saturation in the steel, should decrease with the carbon content. The increasing percentage of pearlite should result in a smaller angular displacement of the magnetic molecule under

the influence of the magnetic field, thus decreasing the magnetic intensity. A further decrease in the intensity should be produced by reason of the smaller number of free ferrite molecules as the pearlite approaches the eutectic percentage. The intensity of magnetization should increase with the percentage of carbon in excess of the eutectic on account of the breaking up of the solid pearlite by the excess cementite, thus freeing some of the ferrite molecules and allowing them larger angular displacement in the magnetic field.

To examine this point the writer has made use of some data taken by Dorsey,¹ to whom these alloys were loaned for a study of the magnetostriction. In the magnetostriction experiments the intensity of magneti-

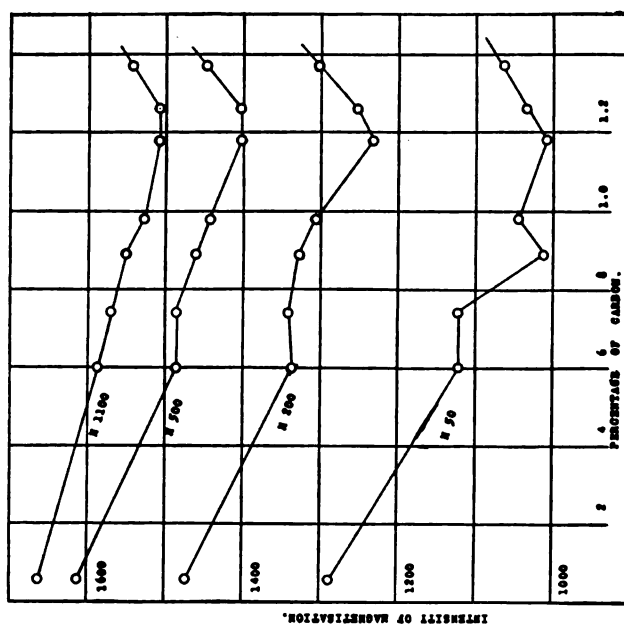


Fig. 3.

Showing the relation between the intensity of magnetization at various field strengths and the carbon content.

zation was determined ballistically, proper corrections being made for the short length of the test bars. By interpolation from the data given, the intensity of magnetization was found for each alloy at the following fields: 50, 200, 500 and 1,100 C.G.S. Fig. 3 shows the curves of intensity of magnetization when plotted against the carbon content.

It will be seen that the minimum of the curves in Fig. 3 corresponds very well with the maximum of the curve in Fig. 1, *i. e.*, near the eutectic

¹ Dorsey, *PHYS. REV.*, Vol. XXX., p. 698, 1910.

percentage. The curves in Fig. 3 further show that the intensity of magnetization at high fields is affected very much less by increasing carbon content than at low fields. In other words, as complete magnetic saturation is approached, the intensity of magnetization is less affected by the chemical composition of the alloy.

The results shown in Fig. 3 are not in agreement with Benedicks,¹ who finds that at a magnetization of 205 C.G.S., the intensity of magnetization decreases only very slightly with the increase in carbon content. There is no indication in his results of a minimum in the curve of intensity of magnetization and carbon content at the eutectic percentage for the steels he studied, but the decrease in intensity of magnetization with the carbon content is very much more marked when the carbon exceeds 1.3 per cent.

THE ELASTIC PROPERTIES AND THE CARBON CONTENT.

The relation between the maximum strength of steels and the carbon content has been the subject of a large number of investigations, but in many cases the results are useless because the heat treatment has not been uniform or because of large variation in the constituents other than the carbon. In order to make this investigation complete a series of

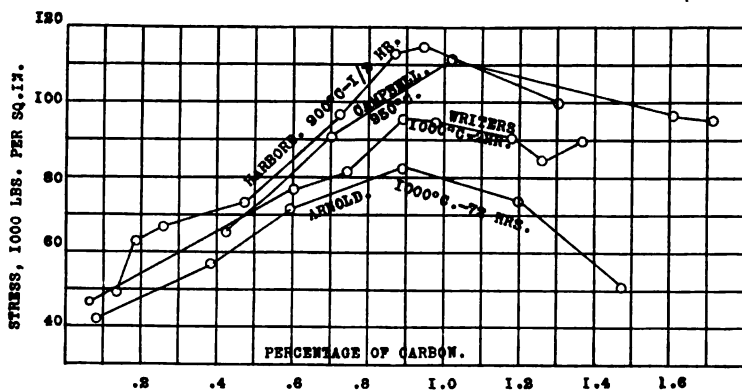


Fig. 4.

Showing the relation between the maximum strength and the carbon content.

strength tests were made on a 20,000 pound Riehle Standard Testing Machine and the complete results were made the subject of another paper.² For purposes of comparison with the magnetic properties, the writer has taken the liberty of reproducing here in Fig. 4, a diagram showing the relation between the maximum strength and the carbon

¹ Benedicks, loc. cit., p. 160.

² Jones and Waggoner, Proc. A. S. for Testing Mat., Vol. XI., 1911.

content. This figure includes, beside the series of alloys under investigation, some other published results.

An inspection of the shape of the curves in Fig. 4 shows a very striking similarity to the curves for the hysteresis loss given in Fig. 1, with the maximum strength at, or near, 1 per cent. of carbon. The difference between the several maximum strength curves is not greater than could be reasonably expected from the difference in the composition and heat treatment. The curves fall in the order in which they would be logically placed by the methods of annealing employed. The particular heat treatment given the steels under test was not chosen to bring out the highest tensile strength, but because these steels had a similar heat treatment for the magnetic tests.

The shape of the maximum strength-carbon curves had been predicted by Howe¹ from the physical characteristics of the microscopic constituents. Ferrite, being soft and ductile has a relatively low strength. As the carbon content increases the ferrite is surrounded by a network of pearlite, thus increasing the adhesion between the crystals. That the strength of unhardened steels reaches a maximum at or near the eutectic is largely due to the small crystallization and very intimate mixture of the crystalline constituents. On the other hand when the eutectic is passed the pearlite is surrounded with a network of cementite and both of these networks have a tendency to destroy the adhesion between the crystals.

It would seem then that the magnetic hysteresis and the maximum strength of steels vary in the same way with changing carbon content; that the similarity between the magnetic and elastic hysteresis is more than a mere similarity of shape of their characteristic curves. That the magnetic and elastic properties are related in some fashion has been known for a long time. Ryder² proposed to classify steels by the residual magnetism of short bars, and Osmond³ in commenting on this said, that in studying this method of classification he had found it in practical accordance with that based on tensile strength. "A great advantage of these magnetic tests," he said, "is that they could be applied to the finished pieces, and not merely to a test-piece more or less restricted as to shape."

A further similarity is shown⁴ in the fact that increasing the density of steel by cooling it in liquid air increases the hysteresis loss when the steel

¹ Howe, loc. cit., p. 162.

² Ryder, Eng. and Min. Jour., 1877.

³ Osmond, Pro. I. C. E., Vol. CXXVI., p. 264.

⁴ Waggoner, loc. cit.

has been magnetized to high inductions, over the loss of hysteresis at ordinary room temperatures; and Hadfield¹ has shown that the low temperature also increases the tensile strength. Hardening steels by other methods results in an increase in both properties; while proper annealing will reduce both factors in the same way.

A comparison of the percentage elongation under stress, or the ductility, when plotted against the carbon content, with the intensity of magnetization when plotted in a like manner, shows that the curves are strikingly similar, indicating that the ductility of these alloys and their intensity of magnetization are affected in the same way by the chemical composition. The maximum susceptibility-carbon curve is also similar to the curve of ductility-carbon, *i. e.*, the maximum susceptibility decreases with increasing carbon until the eutectic is reached and then again increases with the increase in carbon content.

THE MAGNETOSTRICTION AND THE CARBON CONTENT.

Dorsey, in his work on the magnetostriction in these alloys, has shown that the maximum percentage elongation of the bar when plotted against the percentage of carbon is exactly similar to the maximum susceptibility curve. Assuming now that magnetostriction is nothing more than an attempt on the part of the ferrite molecules to orient themselves in a given way under the influence of the magnetic field, the shape of the magnetostriction-carbon curve with its minimum at, or near, the eutectoid, may be explained in a manner similar to that given above for the maximum strength and hysteresis; *i. e.*, it would seem reasonable to expect the smallest percentage elongation in a steel having the eutectic percentage of pearlite, since this percentage of ferrite and cementite is such as to produce the maximum adhesion between the crystals.

SUMMARY.

1. In a series of unhardened steels, the relation between the hysteresis loss, intensity of magnetization, maximum susceptibility and the carbon content might have been predicted from the Ewing theory of magnetism and a knowledge of the physical characteristics of the microscopic constituents of such steels.
2. The magnetic hysteresis loss and the maximum strength of unhardened steels vary in the same way with the percentage of carbon.
3. The intensity of magnetization, at saturation fields, and the maxi-

¹ Hadfield, J. I. and S. Inst., Vol. LXVII., p. 143, 1905.

mum susceptibility show a like variation with carbon content, as the ductility of these steels.

4. The maximum percentage elongation due to magnetostriction in a series of annealed steels seems to be a function of the ductility of the steel and may be explained on a basis of the elastic properties of the microscopic constituents comprising these steels.

PHYSICAL LABORATORY,
WEST VIRGINIA UNIVERSITY,
March 28, 1911.

THE CRYSTALLIZATION OF CARBON-DIOXIDE, NITROUS OXIDE AND AMMONIA.

By H. E. BEHNKEN.

A DISCUSSION by Professor Gill, of the similarity of the chemical compounds of carbon and silicon, during a study of the quartz crystal, suggested the problem of determining the crystallization of the dioxide of carbon.

Several methods for securing crystals were considered and tested.

1. The solution of the gas in a liquid of low freezing point and a microscopic study of the crystals formed on reducing the temperature below that of the sublimation of CO_2 crystals. This was tried with petroleum ether as the liquid, but no satisfactory study of the resulting cloudiness was made.

2. The freezing of the liquified gas.

3. The solution of the gas in some cold liquid of low boiling point and the production of crystals on evaporation.

4. The production of crystals by direct freezing. Two schemes were proposed; either to pass the gas through a very cold liquid, or to allow it to come in contact with a cold solid surface. The latter proved successful.

The chief difficulty was met in the elimination of water vapor, the crystals of which might not be distinguishable or whose presence in quantity would make results uncertain. While this would appear a simple matter, with the powerful dehydrating reagents available, it was not found possible to prevent the formation of some ice crystals.

The essentials of the apparatus, as suggested by Professor Shearer, consisted of a metal cup into which the gas was directed, a microscope for studying the crystals, and a Dewar cylinder of liquid air to be used as the freezing agent; the cooling was controlled by means of a copper rod attached to the bottom of the cup and immersed in the liquid air. By varying the distance between the surface of the cold liquid and the cup, the temperature of the freezing space could be regulated.

Difficulty in illuminating the field on an opaque surface in a Dewar tube caused the substitution of a glass freezing plate. By means of a series of reflections, the light could be sent up through the plate.

An arc light (*A*), Fig. 1, was used as the source of illumination, the heating effect of which was avoided by passing the light through the water cell (*W*). The light was then converged (*C*) and reflected down into the Dewar by the mirror (*M*).

The light in the Dewar (*D*) was reflected at right angles by a prism (*P*) to a small concave mirror (*R*) set in the freezing tube (*T*). This mirror was not actually made, being in fact a piece cut out of a broken silvered Dewar bulb. It was set at approximately 45° as shown, being held in place by pieces of tubing, cut at an angle, and slipped inside the main tube. After this reflection, the light passed through the freezing stage (*S*) to the microscope.

What has been spoken of as the "freezing tube" is a short length of Tobin bronze tubing, about $7/8$ inch outside and $5/8$ inch inside measure. A rabbet was cut around the top to hold the piece of glass on which the crystals were to form; this was a microscope cover glass, cut down to fit. To this tube was connected a brass rod (*B*) which dipped into the liquid air (*L*) and by means of which the temperature could be controlled. A hole was bored in the side of the freezing tube to admit the light to the mirror inside.

To exclude the outside air with its attendant moisture, the freezing plate was enclosed in a chamber made of a four-inch length of $2\frac{1}{2}$ inch thin glass tubing. The bottom of this chamber consisted of a wooden disk, cemented in with shellac, and thoroughly coated with the same. Through this disk, a hole was bored which admitted the freezing tube, a collar soldered to it holding it up. This hole being eccentric, allowed a limited horizontal shifting of the field under the microscope by turning the chamber. To allow of the focusing of the microscope, it was equipped with a fitted wooden disk covered with vaseline (*W*) which, by acting as a piston in the chamber, kept out the air completely. The CO_2 was admitted by a glass tube passing through a hole cut in the glass wall of the freezing chamber; all the space in this hole not occupied by the tube was filled with a lump of vaseline. To aid in keeping this chamber dry, a small boat filled with P_2O_5 was introduced.

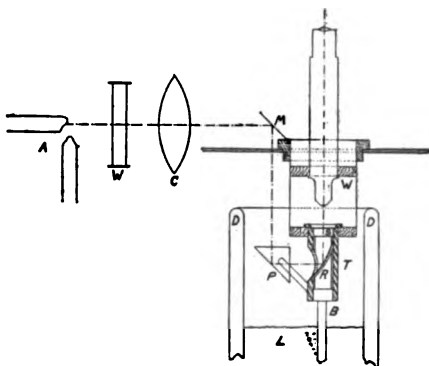


Fig. 1.

A, arc; *W*, water; *C*, condensing lens; *M*, mirror; *DD*, Dewar cylinder; *P*, reflecting prism; *L*, liquid air; *T*, freezing tube; *R*, concave mirror; *S*, glass freezing stage; *W*, wooden disk piston on microscope.

Some difficulty was encountered in keeping the reflecting prism and other surfaces free of frost. This was partially overcome by enclosing the entire system so that the evaporated liquid air, which was practically dry, filled the enclosure. Even with this, frost slowly gathered and the prism must be warmed at intervals.

To allow the raising and lowering of the Dewar without admitting outside air, a square telescope box was used. The entire arrangement

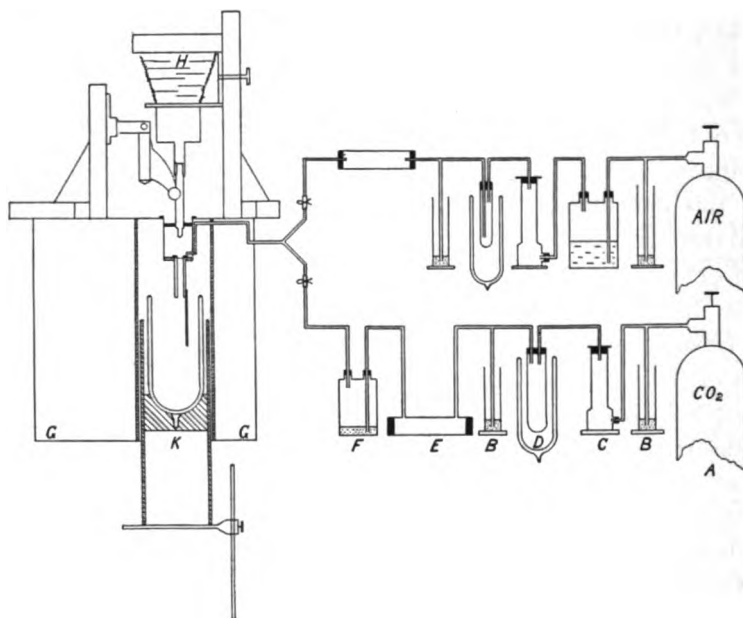


Fig. 2.

(A) Tank of liquid CO_2 . (B) Safety by-passes, T tubes with end dipping below surface of mercury in cylinder. The depth of immersion determines safety point. (C) Calcium chloride drying tower. (D) Test-tube with Dewar tube of liquid air in which it can be immersed to various depths. Dewar tube is on adjustable stand. (E) Drying tube filled with glass wool dusted with P_2O_5 . (F) Bottle with small amount of mercury by means of which the rate of flow of the gas could be watched. (G) Large glass case serving as enclosure and as support for bulk of apparatus. (H) Camera on adjustable stand. (K) Telescoping box holding large Dewar cylinder; supported by stand under table which rests on floor. Inner box of telescoping box passes through hole in table.

was enclosed in a large glass case, which, beside serving as an enclosure, supported the microscope, camera, freezing chamber, and reflector as shown in Fig. 2. The outer, fixed tube of the telescoping box extended from the top of the table to the top of this case, and was equipped with a strip of glass through which to observe the height of the Dewar. The inner, sliding tube, which was closed by the base carrying the Dewar,

passed through a hole cut in the table to fit it, but allow it to slide up and down. A quantity of CaCl_2 in the outer case helped to keep the moisture down. The tube carrying the Dewar was supported by an adjustable iron stand resting on the floor.

Liquid CO_2 in cylinders was allowed to vaporize and the gas was purified by freezing it in a tube immersed in liquid air, on allowing the solid to sublime slowly, the rate being controlled by the depth of immersion, a high degree of purity was secured.

The greatest problem was the elimination of moisture from the gas. This has not been entirely mastered, for ice made its appearance in spite of all precautions. The CO_2 from the tank was first passed slowly through a CaCl_2 drying tower 12 inches high. It was then frozen in liquid air as described above. Before the gas secured by sublimation was admitted to the freezing chamber, it was passed through a 14-inch condenser tube filled with glass wool well filled with P_2O_5 . As an additional precaution a small boat of P_2O_5 was placed in the freezing chamber. But in spite of all precautions, fairly large ice crystals were formed after passing the gas into the chamber for a half hour at a temperature too high to form CO_2 crystals. The fact that long crystals (needles) of this ice grew on top of the layer of phosphoric acid formed over the P_2O_5 in the boat showed the inefficiency of this agent under these circumstances, or when long runs are made. As the temperature rose, the unchanged P_2O_5 absorbed this moisture. Of course this did not occur in the short runs necessary for the production of the CO_2 crystals.

To test the efficiency of the drying arrangement and to make sure that CO_2 was not necessary to the production of this form, a duplicate freezing and drying train was built, through which air purified by lime, solid KOH , and KOH solution was passed. This pure air produced the same crystals, indicating that moisture alone was the source. This train was connected to the other by a Y tube and was useful in flushing the freezing chamber and also, by supplying air during cooling, preventing the drawing in of atmospheric air by contraction of the gas in the chamber.

In forming the crystals, different procedure was found to give great difference in results. If a considerable stream of CO_2 was allowed to flow into the chamber before and during the cooling, the plate would be covered rapidly, when the right temperature was reached, by crystals in fern leaf forms. These would quickly unite and produce a solid covering. If the plate were barely cold enough, they would begin to sublime on shutting off the CO_2 . If the plate was first well cooled, and the stream was turned directly on it, the same forms were produced.

The best results were obtained by delivering the gas tangentially at the edge of the cooling stage. In this way, the gas would be delivered on the plate gradually, although it might come out of the tube at a greater rate. Much of it was frozen on the brass, sometimes producing a transparent glaze. This procedure caused the gas reaching the glass plate to be gradually cooled, maintaining more even conditions and tending to better crystal formation.

The best crystals were obtained by reducing the plate to a moderately cold condition and then admitting the gas in *limited* quantities. A quantity of solid CO_2 was first collected, the non-freezing portions being allowed to escape to the air by one of the safety vents. The solid CO_2 was then allowed to sublime slowly, flushing out the apparatus. The CO_2 was then shut off and the pure air turned into the chamber, sweeping out all residual gas. The Dewar cylinder having been filled with liquid air by means of a tube passing through a hole in the box, was then raised to bring the level of the liquid air about three inches below the freezing plate. The passing of the dry air prevented the drawing in of the outer air on cooling. The air is now shut off, the CO_2 being turned on at the same time. Only a small amount must be admitted at this time, as otherwise too many crystals are formed. They seem to flash into existence. By feeding the CO_2 very slowly or in a series of very small puffs, crystals could be grown to any size. If it is fed too rapidly, crystals with re-entrant angles are produced. After the formation of the crystals, the CO_2 is shut off and the Dewar is raised to cool the plate still further and so prevent sublimation. Often a "snowstorm" of small CO_2 crystals occurs, covering and obscuring the larger crystals and causing a mottled appearance of the pictures. After the temperature had risen high enough to vaporize the CO_2 , small heaps of ice remained which, on further warming, ultimately sublimed in the dry atmosphere or melted when large. None of such size, however, were produced in making the regular runs, as the quantity of gas used was small and the water accordingly minute. Considerable time elapsed between the evaporation of the CO_2 and the disappearance of the water, showing the difference in temperature.

If the entire chamber is too cold, and the CO_2 is turned on, a veritable CO_2 snowstorm occurs which is worthless for our purpose. If the plate alone is very cold and a great puff of gas is turned in, "spiderweb" forms may be produced.

Some clear patches were often observed containing no crystals. These were caused by a sort of explosion. Unfortunately, the eye was drawn to a point only after the explosion was over, so that what had really taken

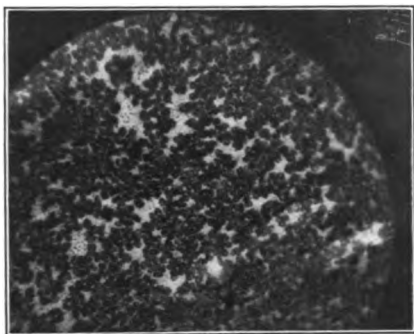


Fig. 3.



Fig. 4.

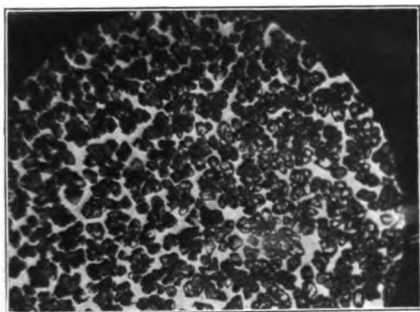


Fig. 5.

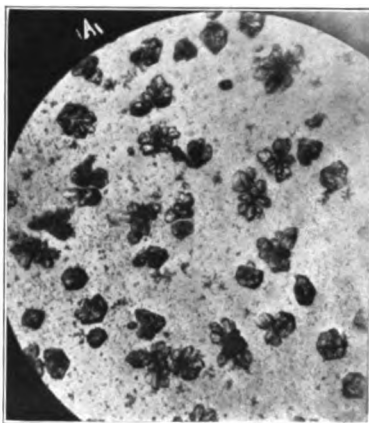


Fig. 6.

H. E. BEHNKEN.

PLATE I.

place could not be learned. It was probably caused by the cooling of a large CO_2 crystal causing its breaking due to contraction. It might be a change of crystal form taking place with a change of temperature. A comparison of three successive plates showed the result of these explosions. Clear spaces in locations where large crystals were shown on plates taken first were noted on the subsequent ones. The sharpest, best-formed crystals seemed most likely to rupture in this manner.

The CO_2 crystals appear to be combinations of cubes and octahedra. Many of the triangular faces with the corners off were clearly seen. Some square faces with the plain triangles are also observed.

On analysis, the gas purified in the way described showed an impurity of 0.16 per cent. unabsorbable in KOH. This percentage has been corrected for vapor tension of water, and calculated on the basis of zero water content of the gas analyzed. The unabsorbed residues of all the determinations were run together and the sum used in obtaining this result.

The work so far described was done in the summer session of 1909 and 1910. The following results were obtained in the summer of 1911.

It was deemed essential to use polarized light in the further study of the crystal forms. A polarizing nicol introduced into the freezing tube below the plate would, in all probability, have broken on cooling. Some compact device that would survive cooling, and if broken could be replaced readily, was needed. A pile of microscope cover glasses ground to fit replaced the concave mirror below the freezing plate. The reflecting prism was moved down and so set that the angle of incidence on the pile of plates was 57° ; the plates being set at an angle of 33° in the tube so as to send the polarized beam directly upward. The rest of the lighting device was unchanged. The analyzer was a regular microscope nicol set above the ocular. A diaphragm made of black paper was used below the first reflector so as to prevent stray light and the inside of the Dewar was coated with dead black to prevent reflection. The freezing chamber and the end of the objective were blackened to avoid reflection of light upon the crystals. The small amount reflected from the object lens could however be noticed.

The darkening of field and CO_2 crystals was simultaneous, and no optical peculiarities could be noted. Some excellent crystals, perfectly clear and with perfect faces were produced.

Professor Gill was kind enough to visit the laboratory and to make a study of the CO_2 crystals. He declared them to be undoubtedly isometric and their forms to be combinations of cubes and octahedra.

USE OF THE APPARATUS FOR OTHER GASES.

Ammonia (NH_3) and nitrous oxide (N_2O)¹ were used because their freezing points were near that of CO_2 .

The N_2O was dried by passing it through P_2O_5 , the same agent being used in the boat in the freezing chamber. The NH_3 was dried by passing it through a tube filled with metallic sodium threads formed by squeezing the metal through a die. The boat was filled with scraped sodium. For this method of drying, I am indebted to Dr. H. B. Browne.

The N_2O crystals formed readily when no moisture was present. When moisture was present or when the gas was fed too rapidly, lint-like forms were produced. On close examination, the threads looked somewhat like coral; on the whole, a formless bunch.

The leaf forms were very easily obtained, but the separate forms photographed were produced only with very slow feeding of the gas after the plate was extremely cold.

Ammonia crystals were very hard to obtain. If the plate were not cooled to a very low temperature the gas first liquified, and then froze into a solid mass. It was found necessary to submerge the freezing tube until the prism and the polarizing plates were almost completely submerged in the liquid air, whose level was not more than 2.5 cm. below the freezing plate. As a result, the polarization was poor and could not be used.

When the pure gas was fed into the cold chamber, a great mass of "ammonia snow" looking much like a mass of cotton was produced. It was very much like the "coral" formed by N_2O . Its method of growth was interesting. A cloud of tiny crystals would appear and fall on the plate. Others would fly into the field and alight on the first comers as though attracted by electric charge. This piling up in single column would go on with occasional branching, each fiber meanwhile swaying about wildly in the field of the microscope.

The crystals photographed were obtained by diluting the dry NH_3 with pure, dry air, obtained from the auxiliary system and feeding this diluted gas slowly into the cold chamber. In both NH_3 and N_2O isometric crystals were obtained. The photographs shown were selected from a considerable number to show the general appearance of the plates. The results obtained with these gases would indicate that with sufficient care in manipulation, the apparatus might be used for determining

¹ N_2O , melting point 102.3° , boiling point 89.8° ; NH_3 , melting point -75.5° , boiling point -38.5° (760 mm.). (Landolt-Bornstein's tables, third edition.)

the crystal form of many gases whose freezing points are only moderately low.

The writer is indebted to Professors Gill, Gage and Chamot for advice, and to Professor Shearer, under whose direction the work was carried out, he wishes to express his gratitude.

CORNELL UNIVERSITY,

August, 1911.

PROCEEDINGS
OF THE
AMERICAN PHYSICAL SOCIETY.

THE EFFECT OF THE CHARACTER OF THE SOURCE UPON THE VELOCITIES OF
EMISSION OF ELECTRONS LIBERATED BY ULTRA-VIOLET LIGHT.¹

BY R. A. MILLIKAN.

SOME time ago² I presented to the Physical Society a brief description of experiments in which the positive potentials acquired by metals under the influence of ultra-violet light were greatly in excess of those obtained by other observers. The results thus far obtained in the further study of this "high-speed" emission may be summarized thus:

1. As in the case of the low-speed emission previously studied the positive potential acquired by the illuminated body appears to be completely independent of the intensity of the source. This result has been tested for intensities which vary in the ratio 15 to 1, but not as yet for intensities of wholly different orders of magnitude (*cf.* below).

2. As indicated in the original report the maximum positive potential acquired, as well as the photocurrent at any potential, is a function of the condition of the illuminated surface, and, in general, can be increased by prolonged illumination. The effects of continued illumination alone have been briefly reported in a recent paper.³

3. The velocity of emission is a function of wave-length, but apparently a different form of function from that found in the case of low speed emission.⁴

4. The velocity of emission is a function of some characteristic of the source other than wave-length and can be pushed up as high as to that corresponding to a positive potential of 500 volts or more by suitable changes in the character of the source of the ultra-violet light.

The present studies have to do only with 1 and 4. High speed emission is obtained only with *spark* sources, and variations in the electrical constants of the spark circuit (capacity, self induction, spark length, energy input) are found to produce large variations in the potential-photocurrent curves. The characteristic difference between the type of curve obtained with a quartz-mercury-lamp source and a spark source is shown in the accompanying figure,

¹ Abstract of a paper presented at the Washington meeting of the Physical Society, Dec. 27-30, 1911.

² *PHYS. REV.*, 30, p. 287, 1910.

³ Millikan and Wright, *PHYS. REV.*, Feb., 1912.

⁴ J. R. Wright, *PHYS. REV.*, August, 1911.

which is one of a large number of curves taken with unanalyzed light. By suitable variation of capacity and self-induction spark-source curves of the type shown in the figure can be made to pass over into curves of the mercury-source type.

In order to determine whether the high velocities obtained with spark sources might not be due to very short ultra-violet rays, or to other sorts of rays not given out by a quartz mercury lamp, the following tests were made. First, the absorption coefficients of the effective rays from both sources were compared and found to be essentially alike. This together with the fact that very thin films of either mica or benzole were found to cut off completely all effects from spark

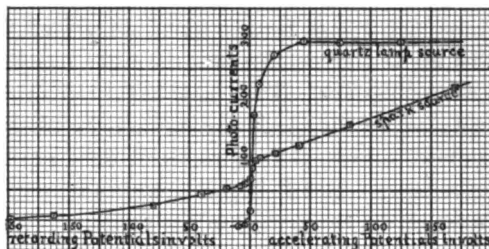


Fig. 1.

sources as well as from mercury lamp sources, indicates that the effective rays consist of ordinary ultra-violet light of wave-length between 260μ and 180μ .

Second, monochromatic light of a given wave-length, obtained with the aid of a quartz spectroscope, shows the same characteristic difference between spark and mercury arc sources as that shown in the figure.

Third, spectroscopic studies of the results of changes in capacity and self-induction with spark sources showed no new ultra-violet lines and no changes whatever in the distribution of the light in the ultra-violet spectrum, while the positive potentials showed enormous changes.

In order to test whether all the differences between spark and mercury-arc sources might not be accounted for by assuming that, while ultra-violet light is necessary to release the electrons, the high velocities imparted to them by spark sources are due to imperfect screening from the intense electromagnetic disturbances of long period which go out from the spark, the following experiments were performed:

First, the whole spark producing apparatus was completely enclosed in a box made of sheet-iron, in one case 5 mm. thick and in another case 2 mm. thick; but this screening produced no essential change in the potential photo-current curves.

Second, the spark and mercury sources were placed so that the light from both could fall simultaneously upon the electrode from which the electrons were to be discharged, then a small screen was inserted so as to cut off the light from the spark source, which, however, was kept running while the mercury source curve was being taken. This was to enable the long electromagnetic waves from the spark to impart high velocities to the electrons liberated by the mercury source provided they were able to do so. *The velocities were in all cases those characteristic of the mercury source alone.* Just as soon however as the screen was removed so as to let the light from the spark, in addition to that

from the mercury arc, fall upon the electrode *the high velocities characteristic of the spark source were obtained*. This seems to show conclusively that the high velocities are due to the light itself, and not to secondary effects of any kind.

The above results have been duplicated for the most part with four different tubes constructed in different ways. Most of the observations too have been repeated by my assistants, Mr. L. J. Lassalle and Mr. A. F. Melcher, for whose help I wish here to make acknowledgment.

In all of the experiments the light has fallen normally upon the electron-emitting surface. This fact, taken in connection with the high velocities produced by the spark source, and the further fact established by the experiments of others, that ultra-violet light projects electrons most readily in the direction in which it is going, makes it easy to account for all the differences shown in the two curves of the figure.

In order to obtain any interpretation of the foregoing high velocities from spark sources it seems necessary either to discard conclusion 1 above, and assume that with such enormous differences in intensity as exist between the light from the mercury arc and that coming at certain instants from a spark source, the velocity of emission of electrons is not independent of intensity; or else to assume that light possesses some other quality besides wave-length and intensity which is responsible for these differences. The latter assumption would be extremely radical. The former is not yet justified by other experimental evidence. If it is permissible to make it, one of the strongest arguments for a "light-unit" hypothesis is removed. In any case these results are completely at variance with the Planck-Einstein light-unit theory in so far as the maximum value of the energy of discharge of electrons by ultra-violet light is concerned.

It should be pointed out that the striking similarity between the curves here obtained with the spark sources of ultra-violet light and the potential-discharge curves obtained by Seitz with the use of X-rays furnishes another indication of an intimate relationship between light and X-rays.

RYERSON LABORATORY,
UNIVERSITY OF CHICAGO.

NOTE ON THE VARIATION FROM LAMBERT'S COSINE LAW OF THE EMISSION FROM TUNGSTEN AND CARBON AT GLOWING TEMPERATURES.¹

BY A. G. WORTHING.

IN the investigation of the conduction losses in incandescent lamps² an unexpected variation from Lambert's Cosine Law was noticed. A special investigation making use of the same method, though considerably refined for this purpose, has been undertaken.

¹ Abstract of a paper presented at the Cambridge meeting of the Physical Society, April 27, 1912.

² Hyde, Cady and Worthing, Trans. Ill. Eng. Soc., 6, p. 238, 1911; Ill. Eng. (London), 4, p. 389, 1911.

For a uniformly heated cylindrical tungsten filament viewed normally to its axis, the variation from the cosine law is such as to make the average brightness for $\lambda = .63\mu$ about 3.0 per cent. greater than that of the central portion; for $\lambda = .46\mu$ the variation is considerably less; for an untreated carbon filament, the variation is such as to make the average brightness about 5 per cent. less than that of the central portion. A change of temperature for the tungsten shows a definite change in the variation from the cosine law. The light emitted from the edge of the carbon filament was found to be about 20 per cent. polarized, that from the edge of the tungsten filament about 60 per cent. polarized, both in planes parallel to the axis of the filaments. These results for tungsten are similar to results obtained by Uljanin for Pt, Ag and Cu.¹

SOME FURTHER RESULTS OF PRECISION MEASUREMENTS OF REFRACTIVE INDICES AS A FUNCTION OF TEMPERATURE.²

By F. A. MOLBY.

THIS paper is a continuation of the work on crown glass, flint glass, and quartz, as reported at the Washington meeting, Dec. 29, 1911. An account of the measurements will be given in a future article in the *PHYSICAL REVIEW*.

ROTATIONS IN THE METALLIC ARC.³

By W. G. CADY.

THE existence of rotations at the anode of the iron arc in air was pointed out in a former paper.³ Further investigation has shown that the following phenomena occur when the arc has burned long enough to settle into a steady state, with a globule of molten iron oxide on the positive terminal. As cathode, iron or carbon have usually been employed, though any other metal may be used as long as it does not cause too great a change in the composition of the positive globule. To keep the globule sufficiently hot, it is best to use as anode an iron rod only a few millimeters in diameter, or else to place the globule on the end of a carbon rod.

At small values of current the positive globule, though liquid, is not in a state of vaporization, and the discharge at the anode is a glow. This constitutes what I have called the "first stage" of the arc.⁴ At about 1.5 amp. the globule commences to vaporize and the current suddenly increases slightly. It is here that the rotations generally begin. If a greatly magnified image of the arc is thrown on a screen, a small bright ring is seen on the anode, indicating

¹ *Ann. der Physik.*, N. F., 62, p. 528, 1897.

² Abstract of a paper presented at the Cambridge meeting of the Physical Society, April 27, 1912.

³ Cady and Arnold, *Am. Jour. Sci.*, 24, p. 383, 1907.

⁴ Cf. Hagenbach and Veillon, *Phys. Zeitschr.*, 11, p. 833, 1910, and Fabry and Buisson, *Jour. Phys.* (4), 9, p. 929, 1910.

that the positive base of the discharge is rapidly describing a circular path. At the same time a very high-pitched whistle is heard.

The arc has been photographed on a rapidly moving film, which shows clearly the rotation of the positive base. Small fluctuations of current and p.d. have been simultaneously recorded with the aid of a high-frequency oscillograph. As the current is further increased the frequency of the rotations increases, reaching a maximum at about two amperes and then falling again, until at three or four amperes the rotations become too irregular to follow. The frequency is usually between 3,000 and 8,000 per second. The rotations can also be seen clearly with the aid of a rotating mirror. They cease in the absence of oxygen in the surrounding gas, and they occur at a smaller current if oxygen is present in excess. These facts, together with the further fact that the positive drop for iron oxide is greater than that for metallic iron, lead to the following explanation of the phenomenon.

If oxygen is present in the surrounding gas, the electronegative oxygen ions are driven to the positive base of the arc, causing the potential drop to increase. The positive base moves aside to a position where there is less oxygen, and thus, thanks to the rounded surface of the globule, it falls into a circular path. The diameter of the path depends on the velocity with which the base moves, and on a central force which prevents its wandering further from its mean position. This central force is furnished in part at least by the tendency of the arc to shorten.

In the equation for circular motion,

$$r = \frac{v^2}{a},$$

r , the radius of the path, and v , the velocity of the positive base, may be observed, and thus the value of the normal acceleration a may be determined, for various values of current and of arc length. It is hoped that the results obtained in this way may throw some light on the conditions prevailing at the anode.

A somewhat similar movement of the positive base is also present, though less clearly, in the case of anodes of nickel or cobalt. Other substances so far tried have failed to show the effect.

In the neighborhood of two amperes, the minute rotations just described often change abruptly to a much larger and slower rotation, of frequency between 700 and 2,000. The ring formed by the positive base is now about a millimeter in diameter, and can be seen directly through a red glass. This form of rotation is more difficult to obtain, and is stable only over a small range of current. When once obtained, however, it is very conspicuous, and it is this form of rotation which was referred to in the earlier paper. It is apparently due to some change—possibly impurities—on the surface of the globule, causing a change in the velocity of the positive base or in the acceleration. A more complete account of this effect will be published later.

WESLEYAN UNIVERSITY.

THE LINEAR EXPANSION OF INVAR STEEL, BETWEEN 100° C. AND -190° C.¹

By F. A. MOLBY.

THROUGH the kindness of Professor J. S. Shearer, a specimen of "Pendulum Steel" has come into the writer's possession, from which he has prepared a specimen 1.366 cm. in length, after the manner of Scheel, Dorsey and others, and, by the method of interference of monochromatic light in vacuum space, the writer has studied the linear expansion of this ring specimen between the temperatures of steam and of liquid air. The maker's description or guarantee of this sample of steel is not known to the writer, but the properties which it is found to have must place it among the high class invar steels. Mr. E. C. Guillaume² gives data for one piece of nickel steel, 36.1 per cent. nickel, which has the equation

$$L_t = L_0(1 + 87.7 \times 10^{-6}t + 12.7 \times 10^{-10}t^2) \quad (1)$$

between 0° C. and +38° C. In another place³ he describes a piece of steel that has a coefficient of expansion so small as not to be measurable with any degree of certainty. The specimen which the writer has used is found to have the following equation between length and temperature, for the range from 100° to -190°:

$$L_t = L_0(1 + 102 \times 10^{-6}t - 22.3 \times 10^{-10}t^2 + 1 \times 10^{-12}t^3) \quad (2)$$

which differs comparatively little from the values given in the equation (1) cited above, but it does differ in that the algebraic sign of the coefficient of the t^2 term is negative instead of positive.

Two sets of measurements were made, giving agreement for the change in length within about two per cent. for the entire range. In the accompanying table are given the mean of the values observed, as scaled from a curve in which all observations were plotted, reckoning from a temperature of zero degrees.

Temp. Range.	Fringes Shift.	δL	$\delta L/L_0$
0° to 20°.0	0.95	0.0000259	0.0000190
0 to 100.	4.08	0.0001090	0.0000799
0 to -100.	- 6.26	0.0001709	0.0001250
0 to -190.	-14.06	0.0003840	0.0002811

The fixed temperatures as steam, room, zero, and liquid air were held constant for such a length of time, one to one and one half hours, that the specimen had entirely come to a steady temperature. The pressures of the air in the optical path through the ring were very nearly one half millimeter, and certainly not so much as one millimeter. The wave-length of light used is 0.00005461 cm., and a shift of one fringe therefore corresponds to a shortening, or lengthening, as the case may be, of the specimen by the amount 0.0000273 cm. That the

¹ Abstract of a paper presented at the Cambridge meeting of the Physical Society, April 27, 1912.

² Comptes Rendus, Vol. 124, p. 178, 1897.

³ Comptes Rendus, Vol. 136, p. 303, 1903.

specimen actually contracts on cooling is known because the fringes shift toward the same edge that they shift toward when the air is being exhausted from the tube in which the specimen is placed. The column entitled δL in the above table gives the change in length for the 1.366 cm. of steel for the corresponding temperature range: the column entitled $\delta L/L_0$ gives the change in length per cm. length for the corresponding temperature range.

In the accompanying curve it may be seen that the length L_t of the specimen

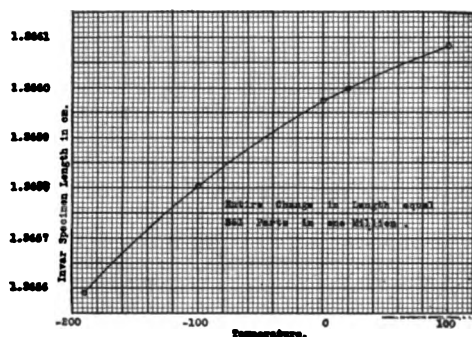


Fig. 1.

is not a linear function of the temperature, but it is quite accurately expressed by the equation (2) given above. A second degree equation might have been determined which would be sufficiently accurate. In the equations of Mr. Guillaume, the coefficient of the t^2 term is positive in all cases noted by the present writer, which would lead us to expect that the effect of temperature should be relatively less at low temperatures, as in

fact it is in the case of the greater number of materials whose low temperature coefficients of expansion have been determined; by the usual way of defining coefficients of expansion as

$$\alpha_t = \frac{dL}{L_0 dt},$$

several values of α may be found by differentiation of equation (2) and by substituting the temperature for which the value of α is desired. In the following table are several such values, dL/dt being the slope of the length temperature curve at the temperature for which the value of α is given.

Invar.		Platinum.	
Temp.	α_t	Temp.	α_t
100	60.4×10^{-6}	100	9.585×10^{-6}
0	102.0×10^{-6}	0	8.749×10^{-6}
-100	149.6×10^{-6}	-100	8.329×10^{-6}
-190	197.6×10^{-6}	-190	8.308×10^{-6}

By comparison with the values derived from Scheel's equation for platinum, it is seen that the invar specimen, although expanding by many times smaller amounts, is not so regular in its expansion as platinum has been found to be.

CORNELL UNIVERSITY,
April, 1912.

¹ Karl Scheel, Deutsch. Phys. Gesell. Verh., 9, pp. 3-23.

THE PHYSICAL REVIEW.

THE HALL EFFECT IN BISMUTH WITH HIGH FREQUENCY CURRENTS.

BY ALPHEUS W. SMITH.

DES Coudres¹ has given a method for determining the Hall effect for alternating currents. The same current flows through the plate and the solenoid which produces the magnetic field in which the plate is placed. If the current in the plate and that in the solenoid are in the same phase, a direct electromotive force is developed at the Hall electrodes and this electromotive force can be measured in the usual way. Since the current in the plate and that producing the magnetic field must be in phase, the circuit must be free from iron and consequently only small magnetic fields can be realized. Observations made by Des Coudres on bismuth by this method indicate that the Hall effect for currents with a frequency of 50 or 700 per second is the same as for direct currents. The work of v. Traubenberg² on bismuth for currents with a frequency of 60 per second leads to the same conclusion. If the Hall effect is caused by the deflection of the electrons in the metal in some such way as the cathode rays are deflected by a transverse magnetic field, the electrons in the plate would always be deflected in the same way, since the magnetic field and the current in the plate alternate simultaneously. If for this or any other reason an appreciable time is required to build up the Hall electromotive force at the electrodes, the method of Des Coudres is not adapted to detect this fact. A method in which the direction of the magnetic field remains fixed and the current in the plate is alternating, so that the Hall electromotive force is also an alternating electromotive force, seems to give a more nearly correct measure of the Hall effect for alternating currents.

¹ *Phys. Zeitschr.*, 2, p. 586, 1901.

² *Ann. d. Phys.*, 17, p. 78, 1905.

Furthermore, Zahn¹ has shown that the method of Des Coudres can not be used for high frequency currents; for it is found that with such currents there are superposed on the Hall effect other effects which are similar to it but more intense. These effects are due to induced currents which, superposed on the primary current in the plate, cause temperature differences between the edges of the plate. These temperature differences set up thermal electromotive forces at the Hall electrodes which behave for the most part as the Hall electromotive forces. The presence of these pseudo-Hall effects make it impossible to determine the true Hall effect by this method.

It was the purpose of the following experiments to determine the Hall effect in bismuth for high as well as low frequency currents by a method which is free from the objections raised against the method of Des Coudres.

To measure the Hall electromotive force with alternating currents a Duddell thermal galvanometer was connected to the Hall electrodes *A* and *B* (Fig. 1). The heater *H* in the galvanometer was of very fine krup-pin wire with a resistance of 3.48 ohms. The leads from the plate to the galvanometer were nearly all of No. 16 copper wire. It was necessary to solder short pieces of somewhat finer wires directly to the plate and then join the leads to these wires. The resistance of the leads amounted to 0.14 ohm. The deflection of the galvanometer is nearly proportional to the square of the current in the heater. The galvanometer was calibrated by sending a current from a storage cell of known electromotive force through the resistance R_1 and the heater. The resistance R_1 was adjusted until the deflection from the direct current was nearly equal to that from the alternating current. The square root of the mean square of the electromotive force between *A* and *B* is given by

$$e = \frac{R_h E \sqrt{d_1}}{R_1 \sqrt{d_2}},$$

where E is the electromotive force of the storage cell; R_h the resistance of the heater and leads; R_1 the resistance in series with the storage cell; d_1 , the deflection of the galvanometer from the Hall electromotive force; and d_2 , the corresponding deflection when the known direct current is sent through the heater. The Hall constant is then given by the usual equation,

$$e = R \frac{Hi}{d},$$

¹ Ann. d. Phys., 36, p. 553, 1911.

where H is the strength of the magnetic field in absolute units; i , the current in absolute units; and d , the thickness of the plate in centimeters.

The high frequency currents were obtained by joining two Duddell arcs X and Y in parallel as suggested by Nasmyth.¹ The currents thus obtained are much steadier than those obtained from a single Duddell arc. The bismuth plate WV in which the Hall effect was to be determined was placed in the secondary circuit which consisted of a self induction L and a capacity K joined to the arcs in the way indicated in Fig. 1. The high frequency currents were measured with a hot-wire ammeter A_2 of the form recently described by Fleming.² It consisted essentially of four fine copper wires in parallel, carrying the current to be measured. A thermal junction soldered to one of these wires was attached to a d'Arsonval galvanometer whose deflection gave a measure of the current flowing in the wire. The ammeter was previously calibrated with direct currents. By varying the capacity and self induction in the secondary circuit currents of different frequencies were obtained.

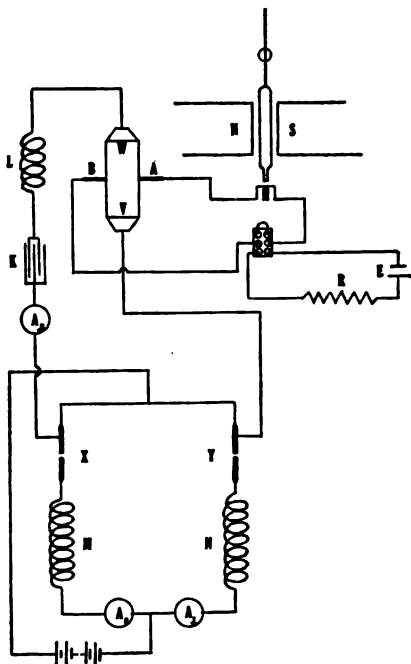


Fig. 1.

Mica condensers were used for the capacities and the frequencies were determined by tuning another circuit which contained a variable capacity and self induction so that it was in resonance with the circuit containing the plate. The frequencies were then calculated from Thomson's formula, $T = 2\pi\sqrt{LC}$.

The plates, which were made from Baker's pure bismuth, were cast in talc moulds in which there had been prepared a cavity of the form of the plate to be cast. From the middle of either side of the plate, which was 4 cm. long, 1.9 cm. wide, and 0.0685 cm. thick, there projected a narrow arm of bismuth 0.25 cm. wide and 1.7 cm. long. These arms served as Hall electrodes and to each of them was soldered a copper wire which led to the thermal galvanometer. To the ends of the plates

¹ *PHYS. REV.*, 28, p. 459, 1909.

² *Principles of Electric Wave Telegraphy and Telephony*, p. 198.

were soldered strips of copper which served as primary electrodes by which the current came into the plate in such a way that the lines of flow were nearly parallel to the edges of the plate. By cutting the arms at their junctions with the plate it was possible to shift them along so as to make them lie very nearly on the same equipotential. Care must be taken to realize this condition or the deflection of the galvanometer will not be a true measure of the Hall effect. The plate was mounted on a piece of red fiber to which it was fastened by a coating of paraffin. Two plates were thus mounted and examined.

In order to have a measure of the accuracy of the method, the Hall constant was first determined for direct currents by the usual potentiometer method. With this direct current flowing in the plate, the Hall electromotive force was then measured by means of the Dudell thermal galvanometer in the manner already described. The magnetic field for these and subsequent experiments was 12,500 gaussses per square centimeter. The observations were made at room temperature. The two methods gave results which agreed within one per cent., an accuracy sufficient for the purposes of this paper. The effect was then measured using alternating currents from an ordinary dynamo of about 60 cycles per second. The values of the Hall constant thus obtained did not differ by as much as one per cent. from that obtained with direct currents under similar conditions.

When the frequency of the currents becomes large the error due to the change in the resistance of the leads to the galvanometer must be taken into account. Fleming,¹ who has studied the change in the resistance of No. 16 copper wire for currents with a frequency of 1.06×10^6 per second, has found that for such frequencies the resistance is 6.62 times its value for direct currents. The value of the resistance for other frequencies can be calculated either from the formula given by Russell² or from that given by Kelvin.³ When this change in the resistance of the leads is calculated it is found that it introduces an error of less than one per cent. for currents with a frequency of 30,000 per second; an error of about one and a half per cent. for currents with a frequency of 50,000 per second; and an error of about four per cent. for currents with a frequency of 120,000 per second. The self induction of the galvanometer leads would also introduce an error in the results. The self induction of the galvanometer circuit was calculated approximately from the formula given by Rosa and Grover.⁴ By comparing the

¹ Proc. Phys. Soc. London, 33, p. 103, 1911.

² Proc. Phys. Soc. London, 21, p. —, 1909.

³ Jour. Inst. Elect. Eng., 18, p. 35 (1889).

⁴ Bull. Bureau of Standards, 8, p. 152 (1912).

impedance of the galvanometer circuit with its ohmic resistance it is possible to estimate the magnitude of this error. It was thus found that when the frequency of the current was 120,000 per second, the observed value of the Hall constant would be two per cent. too low. For the other frequencies used here the error is less than one per cent. The influence of the capacity of the leads was small enough to be neglected. The necessary correction for the sources of error just pointed out has been made in the values of the Hall constants given in Table I. The uncorrected values have also been inserted in brackets.

TABLE I.

Plate.	Method.	Current in Amps.	Frequency.	<i>R</i>
<i>A</i>	Potentiometer	3.58	0	2.77
<i>A</i>	Duddell Galv.	3.58	0	2.75
<i>A</i>	Duddell Galv.	3.60	60	2.75
<i>A</i>	Duddell Galv.	3.45	30,000	2.73 (2.73)
<i>A</i>	Duddell Galv.	4.00	50,000	2.74 (2.70)
<i>A</i>	Duddell Galv.	3.65	120,000	2.74 (2.59)
<i>B</i>	Potentiometer	3.50	0	2.47
<i>B</i>	Duddell Galv.	3.50	0	2.46
<i>B</i>	Duddell Galv.	3.48	60	2.46
<i>B</i>	Duddell Galv.	3.60	30,000	2.47 (2.46)
<i>B</i>	Duddell Galv.	3.80	50,000	2.50 (2.47)
<i>B</i>	Duddell Galv.	3.60	120,000	2.47 (2.33)

An inspection of this table shows at once that within the error of these observations the Hall constant is not a function of the frequency of the currents. Of these values of *R*, the one for currents having a frequency of 120,000 per second is the most uncertain because of the larger corrections which were used in calculating it. These corrections, which together amounted to about six per cent., were calculated with an accuracy such that the tabulated value of *R* does not differ by more than two per cent. from the true value. The error in the other values of *R* does not exceed one per cent.

PHYSICAL LABORATORY,
OHIO STATE UNIVERSITY.

SOME ASPECTS OF THE DYNAMICS OF THE GYROSCOPE.

BY FREDERICK SLATE.

BESIDE the algebraic simplicity due to selecting principal axes, which was probably Euler's single conscious aim, his dynamical equations for a rigid body present a notable peculiarity that he was less in a position to recognize completely. If the equations are written

$$M_1 = A \frac{d\omega_1}{dt} + (C - B)\omega_2\omega_3, \text{ etc.}, \quad (1)$$

the angular acceleration happens to be given accurately by the relation

$$\dot{\omega} = \frac{d\omega_1}{dt} + \frac{d\omega_2}{dt} + \frac{d\omega_3}{dt}, \quad (2)^1$$

making it evident to inspection that the effects of applied force-moment are here segregated on an essentially kinematical basis. This results naturally from the trend of the original demonstration whose leading thought attaches to angular velocity and its changes;² and our compact vector methods allow us to parallel and complete Euler's analysis in a few lines. For any element located by its radius-vector from the center of mass, the relative velocity, acceleration and force are

$$v = [\omega r]; \quad \dot{v} = [\dot{\omega} r] + [\omega v]; \quad dm\dot{v} = dm[\dot{\omega} r] + dm[\omega v]. \quad (3)$$

The corresponding total force-moment required is

$$M = \int dm[r\dot{v}] = \int dm[r[\dot{\omega} r]] + \int dm[r[\omega v]] = M' + M''. \quad (4)$$

Then simple routine transformation of the integrals shows for the components of M' and M'' respectively,

$$A \frac{d\omega_1}{dt}, \text{ etc.}; \text{ and } (C - B)\omega_2\omega_3, \text{ etc.};$$

establishing in the two types of acceleration the basis of separation in the second member of eq. (1).³

¹ In the absence of a standardized notation, some reliance must be placed upon the context, where there are transitions back and forth between vectorial and algebraic statements, in order to prevent confusion. Keeping this in view, the intended sense of the equations here should become clear.

² Euler, *Theorie der Bewegung* (German trans. by Wolfers, 1853), pp. 431-443.

³ This interpretation of the first term (M') is a corollary of a theorem announced by Minchin, *Nature*, Vol. 23, p. 62 (1881). The source of the second term (M'') did not escape Euler himself (*loc. cit.*, p. 323).

But since Euler's day it has been discovered how the idea really at the root of his original plan can be extended. In fact the propositions covering the more general use of so-called "moving axes" have been standard material for at least thirty years; it is therefore somewhat remarkable that they have been so neglected in actual application (except to the special case) until their more recent vector statement supplied a renewed stimulus.¹ It has become a familiar truth now, however, that with the usual assumption about symmetry (*i. e.*, $A = B$) in the gyroscope, we can retain Euler's scheme to the extent of continuing to use principal axes, without being limited by the condition underlying eq. (2). In other words the lines chosen to project upon may have any differential shift about the figure-axis, relative to the body itself. This plan offers an advantage in its more inclusive view, and does not deserve to be so nearly ignored as an alternative; thus the considerations that follow may be taken for a supplement to the "orthodox" presentation. At the same time, because they abandon the lines of kinematical description, and lean at some vital points rather toward directer dynamical statement, they foster a tendency to be encouraged.

We are to conceive the problem of the gyroscope in the usual way: A rotating rigid body having "universal-joint freedom" round a fixed point (O) is under control of weight-moment. With origin at this fixed point assume the following right-handed set of principal axes (possible when $A = B$): The figure-axis (C) with a perpendicular (B) to it in a vertical plane; and a third axis (A) normal to that plane. The resultant force-moment (M) is then always being exerted about the instantaneous position of the line (A), which therefore coincides also with the vector derivative of the moment of momentum (Mm). This implies for components and resultants; the latter being of necessity the actual "physical values":

- (a) $\omega_1 + \omega_2 + \omega_3 = \omega$ (angular velocity),
- (b) $A\omega_1 + B\omega_2 + C\omega_3 = Mm$ (moment of momentum),
- (c) $M_1 + M_2 + M_3 = M = M\dot{m}$ (external force-moment).

Further we adopt the standard angular coördinates:

- ϑ [measured from the downward vertical (Z) to the figure-axis],
- ψ [specifying the azimuth of the plane (ZOC)],
- ϕ [giving displacement relative to (ZOC) about (OC)].

When a progressive shift of axes is supposed a difference is introduced thereby between the two time-rates

¹ The main proposition did not find a place in the earlier editions of Routh's Dynamics; and though Klein and Sommerfeld (*Theorie des Kreisels*, p. 113) emphasize Hayward's exhaustive treatment of the matter (1854), still their equations are exclusively of the classic type (where Cartesian) subject to the condition of eq. (2).

$$M\dot{m} \text{ and } \frac{d}{dt} \{A\omega_1 + B\omega_2 + C\omega_3\} \equiv \frac{d'}{dt} Mm,$$

and gives occasion for a corrective relation that may be written

$$M\dot{m} = \frac{d'}{dt} Mm + [uMm]; \quad (5)$$

where without breaking away from principal axes we can have

$$u = \psi + \vartheta + k\dot{\varphi}; \quad (6)$$

and Euler's equations reappear for $k = +1$. For application to present conditions $k = 0$; so that the difference of plan disappears when $\dot{\varphi} = 0$. Inserting the special value of (u) ; also $A = B$, $\omega_1 = \dot{\vartheta}$; and anticipating the constant magnitude of $(C\omega_3)$, we find

$$M = M_1 = \frac{d}{dt} \{A(\dot{\vartheta} + \omega_2)\} + [\dot{\vartheta} + \psi, Mm]. \quad (7)$$

Without going into details we may remark that this expression is well adapted to a perspicuous grouping of important particular cases; the ordinary weight-pendulum, the spherical pendulum, the adjustments to regular precession. It is favorable besides that M_2 and M_3 are both zero always. Thus the Cartesian expansion of eq. (7) falls at once into the form

$$-mg\bar{r} \sin \vartheta = A \frac{d^2 \vartheta}{dt^2} + \psi \sin \vartheta C\omega_3 - \psi \cos \vartheta A \dot{\psi} \sin \vartheta, \quad (8)$$

$$0 = A \frac{d\omega_2}{dt} + \psi \cos \vartheta A \dot{\vartheta} - \dot{\vartheta} C\omega_3, \quad (9)$$

$$0 = C \frac{d\omega_3}{dt} + \dot{\vartheta} A \dot{\psi} \sin \vartheta - \dot{\psi} \sin \vartheta A \dot{\vartheta}. \quad (10)$$

These only elaborate the one central fact that no vector change in moment of momentum is ever being produced, except the increment along the instantaneous position of the A -axis. The scheme of projection follows up this sole positive controlling action by means of eq. (8); from which in the equivalent form,

$$-mg\bar{r} \sin \vartheta = \frac{d}{dt} (A\dot{\vartheta}) + [\psi Mm_{BC}], \quad (11)$$

we see at once that the external force-moment is absorbed completely into two effects: (1) Changed azimuth of moment of momentum found in the plane (BOC) ; (2) changed magnitude of the horizontal part $(A\dot{\vartheta})$. Any further changes therefore are in character internal readjustments; which must moreover harmonize with the absence of control expressed

by the first members of equations (9) and (10). The possibility of such automatic interchanges ceases to be puzzling when we remind ourselves that vector constancy is not in general consistent with constancy of projections upon lines subject to angular shift; indeed the vector product in expressions like eq. (5) marks and measures the need of external control when moment of momentum remains in constant relation to shifting axes. For instance, the clearest reading of eq. (9) understands the directional changes recorded in the last two terms to be made at the expense of the associated magnitude-change (or *vice versa*), simply because no external source of moment of momentum is available. The two familiar constant values characteristic of this problem are then not similar in their foundation; the moment of momentum about (*OZ*) is constant because that axis is permanently parallel to the weight, while the axis (*OC*) only happens to be unique among the shifting axes in its self-compensated process of gain and loss, owing to the condition ($A = B$). The changes resulting from external cause and those entailed by internal constraint are evidently assembled in eq. (7), whose general plan in the second member is then seen to be a segregation according to the change in (1) magnitude and (2) direction, occurring in the moment of momentum. Yet plain and simple as the guiding thought here proves to be, it has been found difficult to dispel the inherited confusion in which such forms of statement remain involved, as a consequence of continuing to overlook the real intention of equations of motion; and their confessed limitations; while we shuffle the terms in them with indifference to everything but the formal mathematical validity of the results. The vital importance of clear perceptions on this issue is our justification for dwelling a little upon it.¹

In their primary sense, equations of motion express a quantitative equivalence (equality) between a net total of external agency (cause) applied to a body, and the response (effect) in changes of that body's

¹ We can note the "physical inversion" implied in Euler's "Zentrifugalmoment" without serious abatement in our appreciation of his inventive genius and general soundness. Nor are we moved to find fault with Coriolis writing in 1835, when he formulates his particular discovery in terms of "Force centrifuge composée." But we must experience disappointment on encountering in the best modern book on the gyroscope these survivals from the early tentative period still masquerading as pseudo-causes; and imbedded incongruously enough in a treatment whose main lines follow Kelvin's lucid dynamical thought. It is psychologically interesting to observe the persistent perversion due to so slight a turn as writing eq. (1) first

$$M_1 = A \frac{d\omega_1}{dt} - (B - C)\omega_2\omega_3,$$

and then

$$M_1 + (B - C)\omega_2\omega_3 = A \frac{d\omega_1}{dt}.$$

state of motion. Conceived in such terms, this "equality of cause and effect" introduces no metaphysical obscurity; nor is it an obstacle to flexible adaptation in the later stage of calculation; but when accurately considered, it reveals two cardinal limitations in the scope of the equations. First the equality subsists between two summations which are only equivalent in their net totals and not identical or interchangeable piecemeal; as eq. (4) or eq. (8) exemplifies. And secondly the constituent items of each summation can be varied indefinitely without disturbing those totals. The reduction to resultant force and force-moment being founded essentially on the work-equation, we recognize readily the substitution of them for the actual distribution of force as being on a basis of limited equivalence. And this is matched in the second member by a wide range in the admissible plans of describing the effects. Both sides of the equation of motion being in these respects artificialized for convenience, we must not seek to extract from either in general a "one to one correspondence" with all details of the physical conditions. The occurrence of a term in the description implies nothing conclusive about a corresponding physical action; and every suggestion becomes illusory when we obliterate the distinctions through original entry under the two rubrics of cause and effect. So the few forms found helpful toward a natural interpretation become conspicuously prominent. The terms of eq. (4) accord with one most fundamental idea: That a rigid body distributes automatically the applied external force, assigning its quota to each mass-element, calculable in terms of local acceleration. The principle illustrated in application by eq. (8) covers the direct expression of force-moment through changes in moment of momentum. But the process of superposition must not be pushed beyond the limits of its validity. This question regarding eq. (10), for example, is empty of content: Do the last two terms assert that these specific elements of force-moment are present; or is this merely a roundabout method of effectively denying the presence of any such elements? Let us accept explicitly these limitations as they may have bearing upon the matters that follow, where an attempt is made at adding dynamical significance to some results that are ordinarily left in the stage of algebraic proof.

Return to eq. (8) and recast it into the form

$$A \frac{d^2\vartheta}{dt^2} = \sin \vartheta \{ -mg\bar{r} - C\omega_3\dot{\psi} + A\dot{\psi}^2 \cos \vartheta \}. \quad (12)$$

As we may see from the discussion of eq. (11), a zero value of this first member entails complete absorption of force-moment in causing purely directional changes of moment of momentum; and then adding the condi-

tion $\dot{\vartheta} = 0$ steadies the motion into regular precession, with the available force-moment exactly adequate to the effects then necessary.¹ This doubly specialized adjustment is seen from eq. (12) to be in every case possible at $\vartheta = 0$ or π for any finite factors within the parenthesis, and commonsense confirms the conclusion; since, though no weight-moment is available, neither is any then necessary, because the vertical (OZ) has become both a principal axis and the axis of rotation. But there remain the adjustments (in a fuller sense) determined by a zero value of the parenthesis itself, with their possible assignment to real and imaginary regions according to the associated parameters in $0 = f(\dot{\psi}, \vartheta)$. Where an imaginary region occurs it is found to separate the two types of real solution, which coalesce when it disappears; so that considerations developed previously for the conical pendulum² apply in parallel here. But the less restricted problem of the gyroscope outruns that parallelism, notably in allowing a fast or a slow precession; two adjustment-values of $\dot{\psi}$ corresponding to elements unvaried otherwise. We shall concern ourselves next with the dynamics underlying this possibility, whose equation of condition has been presented in the two forms:³

$$\dot{\psi} = \frac{C\dot{\varphi} \pm \sqrt{(C\dot{\varphi})^2 + 4mgr\bar{A} \cos \vartheta}}{2(A - C) \cos \vartheta}$$

and

$$\dot{\psi} = \frac{C\omega_3 \pm \sqrt{(C\omega_3)^2 + 4mgr\bar{A} \cos \vartheta}}{2A \cos \vartheta}. \quad (13)$$

Ordinarily the comparison of the roots ($\dot{\psi}_1, \dot{\psi}_2$) is best attached to the second form, on account of the constant (and therefore common) value of $(C\omega_3)$. The other seems to seek a mathematical purity in being formally explicit, whereas $C\omega_3 = C(\dot{\varphi} + \dot{\psi} \cos \vartheta)$. But this advantage is illusory in practice, if we are obliged to make allowance for variation in $(\dot{\varphi})$ in order to collate the two pairs of roots under most natural conditions. It will fit our present purpose, however, to note the emphasis of the first form in so far as it follows the superposition

$$\omega = \dot{\psi} + \dot{\varphi} + \dot{\vartheta}, \quad (14)$$

and the building up of moment of momentum by stages that match.

¹ Eq. (11) follows the thought more closely, if we suppress the first right-hand term. Such "equations of adjustment" (type $Q = R$) are different from "equations of equilibrium" (type $Q + R = 0$), however slight the mathematical barrier between them. This is sometimes overlooked; zero = parenthesis of eq. (12) being classed erroneously as a "condition of equilibrium."

² Slate, *PHYS. REV.*, Vol. 21, p. 166.

³ The first is apparently peculiar to Klein and Sommerfeld (*loc. cit.*, p. 178) among the prominent writers on the subject.

If positions of the figure-axis on the boundary of a quadrant are excluded, the vertical is not a principal axis; and consequently at the first stage the moment of momentum comprises a horizontal part ($H = \psi(A - C) \sin \vartheta \cos \vartheta$) in the plane (ZOC) as well as the vertical part ($V = I_s \dot{\psi}$). Stopping at precession-adjustment, the scheme is completed by adding the component round the figure-axis ($C\dot{\varphi}$), and the alternative roots going with a common value of ($C\omega_0$) must in view of eq. (11) satisfy the relation

$$\begin{aligned} -mgr \sin \vartheta_1 &= [\psi_1, I_s \dot{\psi}_1 + H_1 + C\dot{\varphi}_1] = [\psi_2, I_s \dot{\psi}_2 + H_2 + C\dot{\varphi}_2]; \\ &= [\psi_1, H_1 + C\dot{\varphi}_1] = [\psi_2, H_2 + C\dot{\varphi}_2]. \end{aligned} \quad (15)$$

On the mathematical side, therefore, this double possibility of adjustment implies merely a numerical equivalence of vector products consistent with certain variations in the factors; though our physical analysis may profitably go one step further. A single type of case will illustrate the point of view sufficiently; it is not necessary to exhaust the algebraic combinations; so let us choose the factors under the radical in eq. (13) all positive, making the roots opposite in sign. Assume $\psi_1 > 0$; $H_1 > 0$; beside $C\omega_0 > 0$. This gives $\psi_2 < 0$; $H_2 < 0$; $|\psi_1| > |\psi_2|$; $|H_1| > |H_2|$; $\dot{\varphi}_2 > \dot{\varphi}_1$; and both values positive in the last inequality agrees with ratios of magnitude found in actual problems. Then the conventions adopted already, with the standard rule governing vector products, lead for the two roots to the effective signs:

$$-mgr \sin \vartheta_1 = \begin{cases} -[\psi_1 H_1] + [\psi_1, C\dot{\varphi}_1] \\ -[\psi_2 H_2] - [\psi_2, C\dot{\varphi}_2]. \end{cases} \quad (16)$$

In connection with the relations of absolute magnitude indicated above, eq. (16) exhibits clearly how the same total force-moment can be partitioned differently between constituents at the two adjustments; so this at once enforces the ideas immediately preceding eq. (12) and also justifies an interpretation, valuable notwithstanding some limitations there set forth, which takes its place alongside those attached to eq. (4) and eq. (7). What reappears everywhere is the subjective core of the descriptive process; reconciling simultaneous constancy and change (by superposition), and treating different sets of components, each in its turn, as equally real content of the resultant.

Again we make eq. (8) the starting-point, this time in order to observe in its first member that the total work is being done continually about the A -axis, while its second member plainly involves the exhibition of resulting kinetic energy elsewhere than in connection with the ϑ -coordinate. The fact that work can be thus "transferred between co-

ordinates," even though they are perpendicular, is at the root of the dynamic stability which becomes so prominent in the gyroscope; and the completeness or the rapidity of such transfer enables us in a way to measure that form of stability or to set the limits of its range. The expression for the total kinetic energy is

$$E = \frac{A}{2} (\dot{\vartheta}^2 + \omega_2^2) + C \frac{\omega_3^2}{2}; \quad (17)$$

and the last term being constant, the variations or interchanges consequent upon work done are confined to the two other terms. Now referring to eq. (9) it is apparent that the initiative (so to speak) centers in the component that is in the line of the external force-moment. So long as $\dot{\vartheta} = 0$ no change can occur in (ω_2) ; but the vanishing of (ω_2, ω_3) separately or simultaneously does not prevent changes in $(\dot{\vartheta})$. This discriminates effectively between stable equilibrium and the "stability" here; the latter depends vitally upon the entrance of (at least) incipient displacement, and is in this respect similar to the "stoppage of motion" (necessarily incomplete) by eddy currents. This feature of gyroscopic mechanisms is always emphasized—that their efficiency is nullified by removing the essential "degree of freedom."

But the influence of the variable components $(\dot{\vartheta}, \omega_2)$ is nevertheless mutual; for we see from eq. (12) that the first member may be made positive or negative, with weight-moment given, by due assignment of the other elements. And in the next place it is clear that the absorption of kinetic energy *away from* the ϑ -coördinate will be more rapid; and therefore completed within smaller displacement if continued; in proportion as that first member has large absolute magnitude, while the first time-rate of $(\dot{\vartheta})$ and the second are opposite in sign. Hence a change of sign in the second time-rate is a critical boundary between conditions favorable and those unfavorable to that process of transfer (*i. e.*, to stability); and the critical equation marking the limit of favorable inequalities is

$$0 = -mg\bar{r} \sin \vartheta - C\omega_2 \dot{\vartheta} \sin \vartheta + A\dot{\vartheta}^2 \cos \vartheta \sin \vartheta. \quad (18)$$

By way of illustration, apply this line of thought to a case of central importance; the examination of stability for a gyroscope with figure-axis pointing vertically upward, and passing through that position ($\vartheta = -\pi$) with angular velocity $\dot{\vartheta} > 0$. Then if this angular velocity is not to increase, nor even to persist, but is to be vigorously "nipped in the bud," $d^2\vartheta/dt^2$, though accurately zero at $\vartheta = -\pi$, must become (strongly) negative at departure from the vertical, and remain negative throughout

a sufficient interval. For $\vartheta = -\pi + \Delta\vartheta$, this leads obviously to the inequality, if we can assume $\cos \vartheta = -1$,

$$0 > +mg\bar{r} + C\omega_3\dot{\psi} + A\dot{\psi}^2. \quad (19)$$

But at coincidence of figure-axis and upward vertical there is a well-known relation, taking the form for our conventions,

$$\dot{\psi} = -\frac{C\omega_3}{2A}; \quad (20)$$

and using this we may simplify inequality (19) into either

$$(C\omega_3)^2 > 4Amg\bar{r}; \text{ or } A\dot{\psi}^2 > mg\bar{r}. \quad (21)$$

In rough summary the lesson here amounts to this: The demands of the directional changes in moment of momentum must develop more than rapidly enough to monopolize the available force-moment as the figure-axis leaves the vertical. Of course the same method can be adapted to less special values of (ϑ); and it can be shown in detail that the energy-relations conform to the requirements outlined.¹ The present purpose is attained in suggesting these aspects of the phenomena. But as a closing word it may be well to remark that the "transfer of work" here spoken of is far from being peculiar to the gyroscope; and it has nothing fundamentally mysterious about it, being introduced as a possibility whenever the resultant force changes direction; as the resultant force-moment does here. It is instructive to consider in parallel the trajectory (weight being regarded as a constant vector) and a planetary orbit at the perihelion. The latter shows the entire kinetic energy gained since passage through the aphelion in connection with a direction instantaneously perpendicular to the resultant force. And with any central force, too, a certain constancy of moment of momentum is dominant, as it is in the gyroscope.

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¹ The point of view alone is claimed as novel. For the results, cf. Klein and Sommerfeld (loc. cit., pp. 249, 321) where they are extracted from the analytic discussion of the graph of a cubic equation.

THE ELASTIC PROPERTIES OF BISMUTH WIRES.¹

BY J. E. HARRIS.

A MOST striking deviation from the laws of elasticity was first observed a few years ago by Guthe² and investigated by him³ and by Sieg⁴ in wires of platinum-iridium alloys. They found that the period of torsional vibration for these wires was by no means constant, but that it decreased rapidly with the amplitude of vibration. The greatest percentage of decrease was found in the smaller amplitudes, the periods seeming to approach a maximum constant value at the larger amplitudes. The logarithmic decrement, starting with the large amplitude, at first increased slowly to a maximum value and then decreased much more rapidly as the amplitude of vibration decreased. The previous treatment of the wire seemed to influence its behavior greatly. After allowing the wire to rest for a long time, or after annealing the wire by heating it to a red heat by means of an electric current, the peculiar properties were not nearly so pronounced as when the wire had just previously been vibrated torsionally for some time. It was also observed that when the wire was kept in vibration for some time at a constant amplitude, the period gradually increased until it finally approached a maximum constant value.

The fact that the previous history and treatment of the wire seemed to influence its behavior to such an extent led to the assumption that the effect was in some way due to the molecular structure of the wire. Since the platinum-iridium alloys are very brittle, it was suggested that perhaps other wires having this same quality of brittleness might show something of the same properties. For this reason, it was decided to investigate the behavior of bismuth wires. This was done and as has been announced by Guthe⁵ and the author, the bismuth wires did show the same large decrease in period and logarithmic decrement with decrease in amplitude.

This paper gives the results of the investigations that have been carried on in the past year with bismuth wires.⁶

¹ This investigation was carried out with the aid of a grant from the Elizabeth Thompson Science fund.

² K. E. Guthe, *Proc. Iowa Academy of Science*, 15, p. 147, 1908.

³ K. E. Guthe and L. P. Sieg, *PHYSICAL REVIEW*, Vol. 30, No. 4, 1910.

⁴ L. P. Sieg, *PHYSICAL REVIEW*, Vol. 31, No. 4, 1910.

⁵ *PHYSICAL REVIEW*, Vol. 32, p. 228.

⁶ The bismuth wires used in these investigations were obtained from Hartman & Braun. They were made from pure, electrolytically deposited bismuth.

II. APPARATUS.

The apparatus was similar in most respects to that described by Guthe and Sieg¹ in their investigations with the platinum iridium wires. A large circular scale was used of such dimensions that each degree of amplitude measured about 2 cm. on the circumference of the scale. Each end of the wire was soldered into a brass cylinder. For the purpose of soldering the wires, it was found necessary to mix some bismuth with the solder in

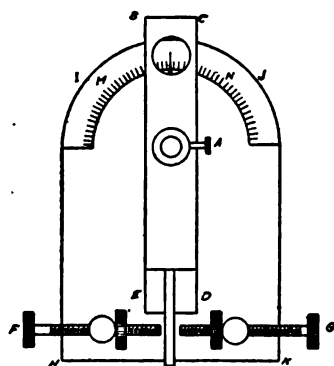


Fig. A.

order to lower its melting point to a temperature below that for bismuth, the melting point of pure bismuth itself being lower than that of the solder. The upper cylinder was clamped in a device shown in the accompanying diagram. The sketch gives a view at right angles to the axis of the wire. The cylinder to which the upper end of the wire is soldered is held in place by the set-screw *A*. The flat piece of iron *BCDE* carrying the cylinder is free to rotate about the axis of the wire as a center of rotation. By pushing this first one

way and then the other, the wire could be made to vibrate. By means of the set-screws *F* and *G*, the angle through which the wire was twisted could be regulated at will. By setting these screws, the wire could also be made, if desired, to vibrate at constant amplitude. The part *HIJK* is a fixed piece of iron and has at one end of it the circular scale *mn*. By means of this scale, the angle through which the wire is being twisted can be determined.

The cylinder attached to the lower end of the wire was turned in such a way as to leave at its lower end a flat disk 5.58 cm. in diameter. Over the stem part could be slipped cylinders of various masses and diameters, so that the mass and moment of inertia of the vibrating system could be varied as desired. A very thin piece of plane parallel glass, silvered on both sides and fastened vertically to the lower surface of the vibrating disk, served for the reflection of light from an arc lamp to the scale. The lower cylinder was hung in a large beaker made of clear glass, the top of the beaker being covered with a piece of cardboard having a hole in the center through which the wire could pass. This arrangement was made necessary, especially in those observations in which a light suspended weight was used, because it was found that the air currents in the room interfered materially with the observations. The beaker was large

¹ Loc. cit.

enough so as not to introduce any damping effect. A cross-wire placed in a beam of light from an arc lamp was focused on the scale after reflection by the mirror by means of a lens.

The periods of vibration were obtained with the aid of a Morse recorder, one of the electromagnets of the recorder being connected to a relay which in turn was connected to a clock beating seconds, the other electromagnet being connected to the apparatus used for determining the time of the passage of the cross-wire through the point of rest. By measuring the distance between the marks on the tape, an accurate determination of the time of each passage of the cross-wire through the point of rest could be obtained.

In the earlier observations, instead of determining the time of passage of the cross wire through the point of rest by means of a tap-key, a selenium cell was placed in the position occupied by the spot of light reflected by the mirror when the vibrating system was at rest. The cell, which was very sensitive, having a resistance of 198,000 ohms in the dark and only 8,000 ohms in direct sunlight, was placed in series with a two-thousand ohm relay and both connected in a 220-volt circuit. The relay was so adjusted that when the light from the mirror passed over the cell in series with the relay, the decreased resistance of the cell would permit enough current to pass to close the relay, which, in turn closed a circuit containing one of the electromagnets of the recorder, thus giving a mark on the tape. For the larger amplitudes, the entire surface of the cell (about $5 \times 2\frac{1}{2}$ cm.) was exposed to the action of the light. However, as the amplitudes of vibration decreased, the spot of light remained for a continually lengthening period of time on the surface of the cell, so that the mark on the tape could no longer be given at the time of the passage of the spot of light over the middle of the cell but before it reached the middle. For this reason, two pieces of card-board were so arranged that they could be closed in from the ends, thus leaving a continually decreasing portion of the cell exposed to the action of the light. It was found that the slit could be narrowed down in this manner until it was considerably less than a centimeter in width. Even this was too wide for amplitudes less than 20° so that when the amplitude had decreased to this point, the records of the passages of light were obtained by means of a tap-key. Unfortunately, due to the passage of too much current through the cell in one of the observations, the sensitiveness was decreased to such an extent that it could no longer be used, and a tap-key had to be substituted in the observations taken thereafter.

III. METHOD OF PROCEDURE.

The object of the first experiment attempted was to determine the way in which the wire behaved as the initial amplitudes of successive observations were gradually increased. For this purpose, a series of experiments was carried on with two different wires, one of .25 mm. diameter and the other .5 mm. diameter, in each case beginning with an observation in which the initial amplitude was small and increasing the amplitude in each succeeding observation. The amplitudes were increased in each case until the elastic limit of the wire was very nearly reached. For all the experiments, a smooth period amplitude curve could be obtained, but when the curves were compared, no definite relation could be obtained. The curves for experiments, in which the initial amplitude was large, would sometimes be above and sometimes below those in which the initial amplitude was small. This was probably due to the different treatment of the wire on different occasions. It was necessary to adjust the relay in series with the selenium cell as the wire was being vibrated in order to bring it up to the desired initial amplitude and sometimes this adjustment was difficult to bring about, especially if the light from the arc happened to be a little variable. For this reason, it was necessary to keep the wire in vibration much longer in some experiments than in others.

From these results, it became evident that some method of procedure would have to be found by which more definite results could be obtained. In Sieg's¹ work, the method was tried of keeping the wire in constant vibration for some time at the maximum amplitude at which it was desired to begin the experiment and then taking a series of observations as the wire came to rest. It was found that if this experiment was repeated at once without the preliminary treatment, the period amplitude curves for the two sets of readings coincided exactly, indicating that the wire, for the time being at least, had been put in a definite state.

A series of experiments were performed to see if in this respect the bismuth wire behaved like the platinum-iridium wire.

1. *Comparison of Two Series of Observations taken One Immediately After the Other.*—In order to determine definitely the effect of the previous treatment of the wire, two series of observations were taken one immediately after the other. In this experiment, a wire was used that had been used in the experiments mentioned above. The wire had, however, been allowed to rest several days before these experiments were performed. The wire was made to vibrate in each case through an initial amplitude of about 85° and the observations taken as the system

¹ Loc. cit.

came to rest. In this and all other experiments with the bismuth wires, we are limited to small amplitudes compared to those used in the platinum-iridium wires, because of the low elastic limit of the wire. By period in this and in subsequent experiments is meant the period of half a complete vibration. The results for the two experiments are given in Table I., *A* being for the first and *B* for the second series of readings.

TABLE I.

Length of wire, 106 cm. Diam., .5 mm. Moment of inertia, 626 g. cm². Temp., 23°. Initial amplitude, 86°.3.

No. of Swing.	<i>A</i>		<i>B</i>	
	Period.	Amp.	Period.	Amp.
5	3.223	73.4	3.224	74.0
10	3.219	63.1	3.221	63.3
15	3.216	54.0	3.217	54.3
20	3.213	46.7	3.215	45.8
25	3.211	40.0	3.213	40.5
30	3.209	34.6	3.210	34.7
35	3.206	30.0	3.207	30.2
40	3.203	26.1	3.205	26.3
45	3.201	22.8	3.203	23.0
50	3.199	19.8	3.201	20.0
55	3.197	17.4	3.200	17.5
60	3.195	15.3	3.198	15.4
65	3.194	13.5	3.196	13.5
70	3.192	11.9	3.194	11.8
75	3.191	10.6	3.193	10.5
80	3.189	9.3	3.192	9.2
85	3.188	8.4	3.190	8.3
90	3.187	7.5	3.189	7.4
95	3.187	6.7	3.188	6.6
100	3.186	6.1	3.187	5.7
105	3.185	5.4	3.187	5.1
110	3.185	4.9	3.186	4.7
115	3.184	4.4	3.185	4.3
120	3.183	3.8	3.185	3.7

The results of this experiment are shown graphically in Fig. 1, the amplitudes being plotted as abscissæ and the periods as ordinates. That the first experiment has had a marked effect on the second is evident from the position of the two curves. The curve for the second experiment lies above that for the first. It will be seen that the second curve in shape is very similar to that of the first. It seems that the effect of the first experiment on the second was merely to displace the curve upward. This effect is due to the so-called "elastic fatigue" in the wire.

2. *Behavior of the Wire when Vibrated at Constant Amplitude.*—In this experiment, several series of observations were taken at different amplitudes to determine whether or not the period remained constant when

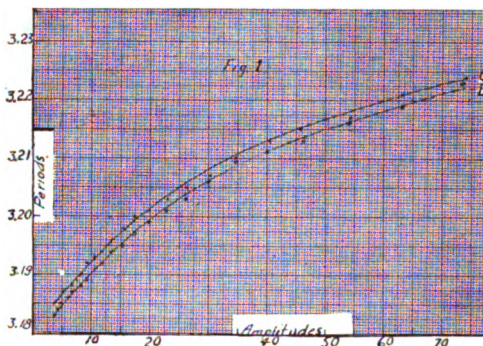


Fig. 1.

the wire was kept vibrating at constant amplitudes. By adjusting the set screws *F* and *G* (see diagram) and moving the lever from one set screw to the other at the end of each swing, the wire could be kept vibrating through any desired amplitude and this amplitude kept constant

TABLE II.

Length of wire = 106 cm. Diam. = .5 mm. Moment of inertia = 626 g. cm².

A. Amplitude = 65°.2.

Coincidence.	Intervals.	No. of Vibrations.	Period.
21.21			
53.42	32.21	10	3.221
108.29	87.08	27	3.225
205.13	183.92	57	3.227

After 20 minutes.

1.56			
85.57	84.01	26	3.231
130.78	129.22	40	3.231
260.07	258.51	80	3.231

B. Amplitude = 93°.5.

37.70			
79.63	41.93	13	3.225
115.12	77.42	24	3.226
192.61	154.91	48	3.227

After 20 minutes.

1.54			
62.96	61.42	19	3.233
224.65	223.11	59	3.233
337.82	336.28	104	3.233

within one or two degrees. Only two of the experiments performed are given here for the reason that all of them gave practically the same results. Two series of observations were taken in each case, the first just after the amplitude desired had been reached, and the second, after the wire had been kept in vibration for 20 minutes. The results are shown in Table II.

It will be seen from the above experiments, that the period of vibration increases when the system is made to vibrate at a constant amplitude; that the period of vibration very rapidly approaches a maximum value; and that this maximum value has apparently been reached after an interval of 20 minutes. To make sure of this latter point, the wire was vibrated for twenty minutes, an observation taken and then the wire vibrated for an hour and another observation taken. The period obtained was the same as that obtained after the twenty-minute interval. That a part of the adjustment of the wire takes place while the wire is being vibrated to bring it up to the maximum amplitude is shown by the above tables. In observation *A*, where the wire was brought to its maximum amplitude by larger steps than in *B*, we have a larger variation in the period in the first 50 vibrations than in the corresponding interval in the second observation.

3. *Effect of Continued Vibrations on the Period Amplitude Curve.*—To show this effect, a wire that had not been used before was taken and after allowing it to hang in place for a day, three series of observations were

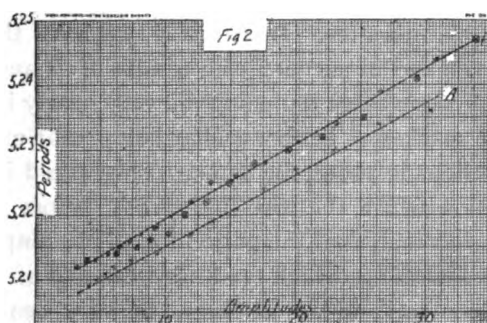


Fig. 2.

taken. In the first, the wire was brought up to the desired amplitude with as few preliminary swings as possible and a series of observations taken as the wire came to rest. In the second experiment, the wire was kept vibrating for a period of about thirty minutes at the same amplitude as that at which the first experiment was begun before taking the series of observations. The third experiment consisted in repeating the first experiment immediately after the second had been completed. The results appear in Table III.

TABLE III.

Length of wire, 103.6 cm. Diam., .25 mm. Moment of inertia, 133 g. cm².

No. of Swing.	Exp. A. Initial Amp. 35°.0. Temp. 20° 5.		Exp. B. Initial Amp. 35°.4. Temp. 20° 5.		Exp. C. Initial Amp. 38°.0. Temp. 20° 8.	
	Period.	Amplitude.	Period.	Amplitude.	Period.	Amplitude.
5	5.236	30.4	5.244	30.8	5.247	33.8
10	5.234	26.3	5.239	26.5	5.241	29.3
15	5.230	22.9	5.234	23.2	5.235	25.4
20	5.227	20.0	5.231	20.1	5.232	22.1
25	5.224	17.5	5.228	17.6	5.230	19.3
30	5.221	15.5	5.226	15.3	5.228	16.9
35	5.219	13.6	5.225	13.5	5.225	14.9
40	5.217	11.9	5.222	11.9	5.222	13.1
45	5.216	10.6	5.220	10.5	5.220	11.5
50	5.214	9.3	5.218	9.2	5.217	10.2
55	5.213	8.3	5.217	8.2	5.216	9.0
60	5.213	7.3	5.216	7.3	5.215	7.9
65	5.213	6.5	5.215	6.5	5.214	7.0
70	5.212	5.9	5.214	5.7	5.214	6.3
75	5.211	5.3	5.213	5.1	5.214	5.6
80	5.210	4.7	5.213	4.6	5.214	5.0
85	5.209	4.1	5.213	4.0	5.213	4.4
90	5.209	3.7	5.213	3.6		
95	5.208	3.3	5.212	3.2		

The results of these experiments are shown graphically in Fig. 2, curves *A* and *B*. In this figure, the amplitudes are plotted as abscissæ and the periods as ordinates. In curve *A*, we have the results of experiment *A* plotted and in curve *C*, the results of *B* and *C*. It will be noticed that in all three cases the points lie very nearly in a straight line. The straight line marked *A* lies somewhat below that marked *B* but the slope of the two lines is about the same. This would indicate that the continued vibration which the wire underwent, before the observations in *B* were taken, has increased the period for all the amplitudes from the maximum to the lowest for which readings were taken. Curve *B* shows that the points for experiments *B* and *C* lie very nearly on the same straight line.

To determine whether or not the results are the same when the wire is kept in continual vibration at a larger amplitude, the experiment was repeated using an initial amplitude of about 190°. This amplitude is slightly under that permitted by the elastic limit of the wire.

The results of these experiments are shown in Fig. 3. The results are quite similar to those found in the preceding experiment. Fig. 3 shows the curve for experiments *B* and *C* lying above that for *A* and the curves for *B* and *C* coinciding. These experiments indicate that the

wire for the time being at least has been put in a definite state. It was determined, as a result of these experiments, that all subsequent series of observations should be taken after the wire had been kept in continuous vibration for a period of from twenty minutes to half an hour.

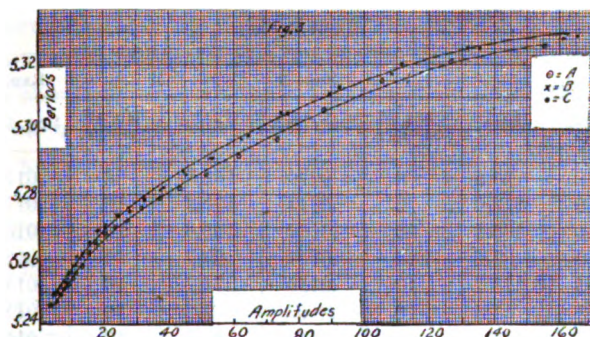


Fig. 3.

TABLE IV.

Length of wire, 106 cm. Diam., .25 mm. Moment of inertia, 133 g. cm².

No. of Swing.	Exp. A. Initial Amp. 144° 8. Temp. 23° 2.		Exp. B. Initial Amp. 190° 3. Temp. 23° 8.		Exp. C. Initial Amp. 195° 0. Temp. 23° 7.	
	Period.	Amplitude.	Period.	Amplitude.	Period.	Amplitude.
5	5.326	156.5	5.328	161.5	5.329	165.8
10	5.321	127.9	5.325	131.7	5.235	135.8
15	5.315	105.5	5.318	108.2	5.320	111.8
20	5.306	87.6	5.311	89.7	5.313	92.2
25	5.297	73.1	5.305	74.7	5.305	76.8
30	5.292	61.3	5.297	62.2	5.298	64.4
35	5.286	51.6	5.291	52.5	5.291	53.9
40	5.282	43.7	5.287	44.2	5.286	45.7
45	5.279	37.2	5.281	37.6	5.282	38.6
50	5.276	31.8	5.279	32.0	5.278	32.9
55	5.272	27.1	5.276	27.3	5.275	28.1
60	5.270	23.3	5.274	23.6	5.274	24.2
65	5.268	20.3	5.271	20.5	5.270	20.8
70	5.265	17.4	5.269	17.6	5.269	17.9
75	5.262	15.2	5.265	15.3	5.265	15.5
80	5.258	13.2	5.261	13.3	5.262	13.5
85	5.256	11.6	5.258	11.6	5.261	11.8
90	5.254	10.3	5.256	10.1	5.258	10.4
95	5.252	9.0	5.254	8.8	5.256	9.0
100	5.251	7.9	5.253	7.8	5.253	7.9
105	5.249	6.8	5.252	6.8	5.251	7.0
110	5.247	6.0	5.251	6.3	5.250	6.1
115	5.247	5.2	5.250	5.6	5.250	5.3
120	5.246	4.6	5.250	4.9	5.249	4.7
125	5.246	4.1	5.249	4.3		
130	5.246	3.6				

The logarithmic decrements for the various amplitudes were worked out for the above experiment and these logarithmic decrements plotted against the corresponding amplitudes in order to determine the effect of continued vibration on the logarithmic decrement. The results are given in Table V.

TABLE V.

Exp. A.			Exp. B.			Exp. C.		
No. of Swings.	Log. Dec.	Amp.	No. of Swings.	Log. Dec.	Amp.	No. of Swings.	Log. Dec.	Amp.
8	.0178	156.5	8	.0179	161.5	8	.0174	165.8
10	.0170	127.9	10	.0174	131.5	10	.0171	135.8
10	.0166	105.5	10	.0167	108.3	10	.0168	111.8
10	.0160	87.6	10	.0161	89.7	10	.0163	92.2
10	.0155	73.1	10	.0159	74.7	10	.0156	76.8
10	.0151	61.3	10	.0153	62.2	10	.0154	64.4
10	.0147	51.6	10	.0148	52.5	10	.0149	53.9
10	.0142	43.7	10	.0144	44.2	10	.0145	45.7
10	.0138	37.3	10	.0140	37.6	10	.0143	38.6
20	.0137	31.8	20	.0136	32.0	20	.0139	32.9
20	.0132	27.1	20	.0132	26.3	20	.0135	28.1
20	.0128	23.3	20	.0130	23.6	20	.0132	24.2
20	.0126	20.3	20	.0127	20.5	20	.0129	20.9
20	.0123	17.6	20	.0125	17.6	20	.0127	17.9
20	.0122	15.3	20	.0124	15.3	20	.0123	15.5
20	.0116	13.2	20	.0121	13.3	20	.0119	13.5
20	.0114	11.6	20	.0120	11.6	20	.0118	11.8
20	.0111	10.3	20	.0116	10.1	40	.0117	10.4
20	.0115	7.9	40	.0108	7.8	40	.0115	7.9
20	.0113	6.0	40	.0106	5.6	40	.0112	6.1
20	.0113	4.6				40	.0109	4.7
20	.0113	3.6						

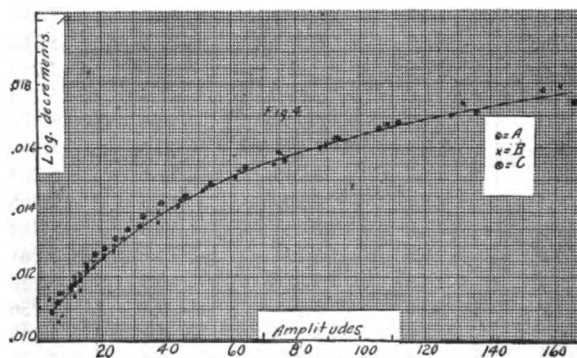


Fig. 4.

In the above table, the first, fourth and seventh columns give the number of swings used in determining the mean logarithmic decrement. The results have been plotted and are shown in Fig. 4. It is seen that

all three sets of points lie very nearly on the same curve, indicating that the continued vibration of the wire has had little if any effect on the logarithmic decrement.

IV. INCREASING AMPLITUDES.

Having settled upon the method of procedure, the first experiment undertaken was to determine the effect of the initial amplitude upon the behavior of the wire. For this purpose, a wire of .25 mm. diameter that had not been used in any experiment before was hung in the bracket with a 50 g. suspension and allowed to hang in position for one day before taking the first series of observations. In the succeeding four days, thereafter, four series of observations were taken, one on each day. On the first, an initial amplitude of $34^{\circ}.4$ was used. On the second the initial amplitude was $84^{\circ}.2$, on the third $131^{\circ}.0$ and on the last $189^{\circ}.5$. In each case the wire was kept vibrating at the initial amplitude for a period of from 20 to 30 minutes before beginning the experiment. The data are contained in Table VI.

TABLE VI.

Length of wire, 103.6 cm. Diam., .25 mm. Moment of inertia, 133 g. cm².

No. of Swing.	Exp. A. Init. Amp. $34^{\circ}.4$. Temp. $20^{\circ}.5$.		Exp. B. Init. Amp. $84^{\circ}.2$. Temp. $22^{\circ}.0$.		Exp. C. Init. Amp. $131^{\circ}.0$. Temp. $22^{\circ}.5$.		Exp. D. Init. Amp. $189^{\circ}.5$. Temp. $23^{\circ}.6$.	
	Per.	Amp.	Per.	Amp.	Per.	Amp.	Per.	Amp.
5	5.244	30.8	5.279	71.9	5.305	112.5	5.328	161.5
10	5.239	26.5	5.272	60.6	5.299	93.3	5.325	131.7
15	5.234	23.2	5.266	51.3	5.291	77.8	5.318	108.2
20	5.231	20.1	5.261	43.7	5.284	65.1	5.311	89.7
25	5.228	17.6	5.256	37.4	5.279	54.8	5.305	74.7
30	5.226	15.3	5.253	32.1	5.273	46.5	5.297	62.2
35	5.225	13.5	5.250	27.6	5.271	39.4	5.291	52.5
40	5.222	13.5	5.247	23.9	5.269	33.7	5.287	44.2
45	5.220	10.5	5.245	20.7	5.263	28.8	5.281	37.6
50	5.218	9.2	5.243	18.0	5.261	24.8	5.279	32.0
55	5.217	8.2	5.241	15.8	5.258	21.3	5.276	27.3
60	5.216	7.3	5.240	13.9	5.254	18.5	5.274	23.6
65	5.215	6.5	5.238	12.0	5.253	16.1	5.271	20.5
70	5.214	5.7	5.236	10.6	5.250	13.9	5.269	17.6
75	5.213	5.1	5.233	9.4	5.248	12.2	5.265	15.3
80	5.213	4.6	5.231	8.3	5.247	10.6	5.261	13.3
85	5.213	3.6	5.229	7.3	5.245	9.4	5.258	11.6
90	5.213	3.6	5.227	6.5	5.244	8.2	5.256	10.1
95	5.212	3.2	5.227	5.7	5.242	7.2	5.254	8.8
100			5.226	5.1	5.240	6.4	5.253	7.8
105			5.225	4.5	5.238	5.7	5.252	6.8
110			5.225	4.0	5.237	5.0	5.251	6.2
115			5.224	3.6			5.250	5.6
120							5.250	4.9
125							5.249	4.3

In Fig. 5 the curves *A*, *B*, *C* and *D* are the period-amplitude curves for experiments *A*, *B*, *C* and *D*. It will be observed that the periods in each successive curve are higher than the corresponding periods of the next preceding curve. These curves are in many respects different from those obtained by Guthe and Sieg with the platinum-iridium wires. In the corresponding experiment with the platinum-iridium wires, while

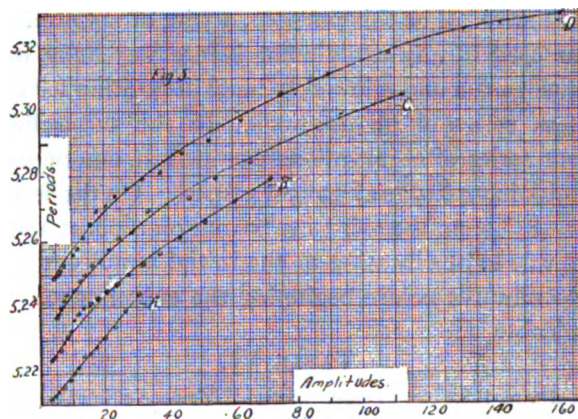


Fig. 5.

the curves for the experiments performed with the large initial amplitudes lay above those for the small amplitudes, the curves seemed to start from a common point, which is not the case with the curves for the bismuth wires. It was also noticed in the experiments with the platinum-iridium wires that at large amplitudes the periods approached a certain maximum value. In this experiment and in all others with the bismuth wires, it was found that, while the curves became less steep with larger amplitudes, no maximum was reached. However, in the case of the bismuth wires, the observations are limited by the low elastic limit to amplitudes that are very low compared to those used with the platinum-iridium wires, so that it may be that the curves shown in the figure correspond to those portions of the curves for the platinum-iridium wires, where the rate of increase in the period is greatest. Also it must be remembered that in the experiment with the platinum-iridium wires the wires were not kept in continuous vibration before taking the observations.

It will be noticed that the curves for the bismuth wires have very much the same shape and slope. The chief difference in the successive curves seems to be one of displacement, the direction of displacement being upward for the larger amplitudes.

The Logarithmic Decrement.—The logarithmic decrements for these

observations were computed and plotted as ordinates with the corresponding amplitudes as abscissæ. The results are given in Table VII. and the curves shown in Fig. 6. The curves *A*, *B*, *C* and *D* correspond to experiments *A*, *B*, *C* and *D*.

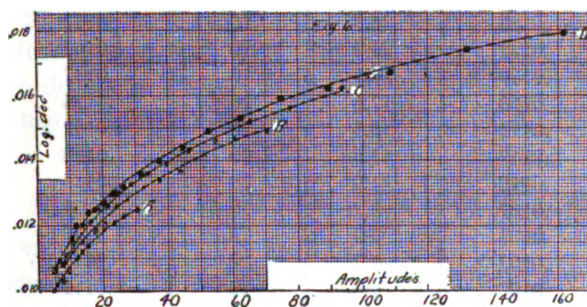


Fig. 6.

It is found that in all the curves there is a marked decrease in the logarithmic decrement with decrease in amplitude. Here, as in the period amplitude curves, we find that each successive curve lies above the next preceding one. These results would indicate that the greater twist of the wire through the large amplitudes increases the internal friction and that this increase in internal friction is not confined to the swings of larger amplitude but persists through the smaller amplitudes.

In the case of the logarithmic decrement-amplitude curves we again notice a striking difference from the corresponding curves for the platinum-iridium wires. In those wires, the logarithmic decrement at first, starting with the large amplitudes, increased slowly to a maximum value and then decreased very rapidly. It may be that here, too, the curve for the bismuth wire corresponds to that portion of the logarithmic decrement curve for the platinum-iridium wires where the amplitudes are small and the values of the decrement are decreasing.

Period and Vibration Number Curve.—For the last of the above four observations, that in which the initial amplitude was greatest, the periods were plotted as ordinates and the corresponding vibration numbers as abscissæ. For lack of space this curve is not shown here. It was found however that this curve also differs from the corresponding curve for the platinum-iridium wire.¹ In the latter case, the curve starts out almost parallel to the *x*-axis, then bends rapidly downward, and finally again approaches a horizontal direction. Neither in this, nor in any other curve of the same sort for the bismuth wire, could the tendency to start out in a direction parallel to the *x*-axis be detected. In other words, the curve is convex downward throughout its length. However, here again, it is

¹ Sieg, loc. cit., p. 425.

TABLE VII.

Exp. A.			Exp. B.		
No. of Swings.	Log. Dec.	Amp.	No. of Swings.	Log. Dec.	Amp.
8	.0125	30.8	8	.0148	70.9
10	.0123	26.5	10	.0147	60.6
20	.0121	23.2	10	.0142	51.3
20	.0119	20.1	10	.0137	43.7
20	.0118	17.6	10	.0134	37.4
20	.0114	15.3	10	.0132	32.1
20	.0112	13.5	20	.0128	27.6
40	.0110	11.9	20	.0126	23.9
40	.0107	9.2	20	.0121	20.7
40	.0103	7.3	20	.0118	18.0
40	.0102	5.7	20	.0115	13.9
40	.0100	4.6	20	.0112	10.6
			40	.0109	8.3
			40	.0106	5.1

Exp. C.			Exp. D.		
No. of Swings.	Log. Dec.	Amp.	No. of Swings.	Log. Dec.	Amp.
8	.0164	112.5	8	.0179	161.5
10	.0162	93.3	10	.0174	131.7
10	.0156	77.8	10	.0167	108.2
10	.0152	65.1	10	.0161	89.7
10	.0146	54.8	10	.0159	74.7
10	.0143	46.5	10	.0153	62.2
10	.0139	39.4	10	.0148	52.5
10	.0136	33.7	10	.0144	44.2
10	.0133	28.8	10	.0140	37.6
20	.0130	24.8	20	.0136	32.0
20	.0126	21.3	20	.0132	26.3
20	.0125	18.5	20	.0130	23.6
20	.0121	16.1	20	.0127	20.5
40	.0120	13.9	20	.0125	17.6
40	.0115	10.6	20	.0124	15.3
40	.0111	8.2	20	.0121	13.3
40	.0109	6.4	20	.0120	11.6
			20	.0116	10.1
			40	.0108	7.8
			40	.0106	5.6

that portion of the curve for the platinum-iridium wires that corresponds to the higher amplitudes that is missing in the curves for the bismuth wires.

In one other respect does the action of the bismuth wire differ from the platinum-iridium wire. In the case of the platinum-iridium wires, it was found that the wire made a larger number of vibrations in coming to rest when started with small amplitudes than when started with

larger amplitudes. Thus when the platinum-iridium wire was started with an initial amplitude of 364.3 degrees, it required 125 swings to fall to 5 degrees, while, when the wire was started with an initial amplitude of 14.3 degrees, 200 swings were required for the amplitude to fall to 6.15 degrees.¹ This same effect can be noted in the bismuth wires but not in anything like the same degree. In experiment *A* above, in which the initial amplitude was 34°.4, it required 89 vibrations for the amplitude to fall to 3.6 degrees. In experiment *D* in which the initial amplitude was 190.5 degrees, it required only 80 vibrations to cover the same interval.

V. EFFECT OF CHANGING THE MOMENT OF INERTIA.

The next experiment undertaken was to determine the effect of changing the moment of inertia of the vibrating system on the period-amplitude curve and on the logarithmic decrement curve. For this purpose, a wire .5 mm. in diameter and 95.7 cm. in length was used. Three experiments

TABLE VIII.

Diam. of wire, .5 mm. Length, 95.7 cm.

No. of Swing.	Exp. A. Mom. of Iner. 1,299 g. cm ² . Mass 150 g. Initial Amp. 89°.4. Temp. 26°.0.		Exp. B. Mom. of Iner. 1,882 g. cm ² . Mass 200 g. Initial Amp. 86°.5. Temp. 26°.0.		Exp. C. Mom. of Iner. 1,456 g. cm ² . Mass 200 g. Initial Amp. 82°.9. Temp. 25°.8.	
	Period.	Amplitude.	Period.	Amplitude.	Period.	Amplitude.
5	4.214	77.1	5.075	74.2	4.471	71.3
10	4.207	64.2	5.068	61.8	4.464	59.4
15	4.200	54.0	5.062	51.6	4.459	49.6
20	4.193	45.3	5.054	43.3	4.453	41.6
25	4.186	38.2	5.048	36.5	4.449	35.1
30	4.181	33.3	5.043	30.8	4.445	29.8
35	4.177	27.5	5.037	26.1	4.441	25.2
40	4.174	23.5	5.033	22.1	4.437	21.7
45	4.171	19.9	5.029	19.0	4.433	18.4
50	4.168	17.2	5.026	16.3	4.428	15.7
55	4.165	14.8	5.024	14.0	4.425	13.5
60	4.162	12.7	5.022	12.1	4.423	11.8
65	4.160	10.9	5.019	10.5	4.419	10.2
70	4.158	9.5	5.015	9.1	4.417	8.9
75	4.156	8.1	5.012	8.0	4.415	7.8
80	4.155	7.1	5.010	6.9	4.413	6.8
85	4.153	6.2	5.006	6.1	4.410	5.9
90	4.151	5.5	5.004	5.3	4.409	5.2
95	4.150	4.8	5.002	4.7	4.407	4.2
100	4.148	4.2	5.000	4.1	4.405	3.9
105	4.147	3.7	5.000	3.7		
110	4.147	3.2				

¹ Guthe and Sieg, loc. cit., p. 617.

were carried out, *A*, with a suspended weight having a mass of 150 g. and a moment of inertia of 1,299 g. cm^2 ; *B*, with a weight of 200 g. and a moment of inertia of 1,882 g. cm^2 ; and *C*, with a weight of 200 g. and a moment of inertia of 1,456 g. cm^2 . By comparing experiments *A* and *B*, it was possible to determine the effect of a change in both the moment of inertia and the mass of the suspended weight. By comparing *B* and *C*, it was possible to determine the effect of changing the moment of inertia only. The results are shown in Table VIII.

The period-amplitude curves are of the same general appearance as those found in the experiments described above. In these experiments, it was found necessary to use smaller amplitudes since the wire had a lower elastic limit than the one of smaller cross-sectional area used in the earlier experiments.

TABLE IX.

B factor = .8813.*A* factor = 1.0615.

Old Period.	New Period.	Amplitude.	Old Period.	New Period.	Amplitude.
5.975	4.473	74.2	4.214	4.473	77.1
5.068	4.466	61.8	4.207	4.465	64.2
5.062	4.461	51.6	4.200	4.458	54.0
5.054	4.454	43.3	4.193	4.451	45.3
5.048	4.449	36.5	4.186	4.443	38.1
5.043	4.444	30.8	4.181	4.438	33.4
5.037	4.439	26.1	4.177	4.434	27.5
5.033	4.436	22.1	4.174	4.430	23.5
5.029	4.432	19.0	4.171	4.427	19.9
5.026	4.429	16.3	4.168	4.424	17.1
5.024	4.428	14.0	4.165	4.421	14.8
5.022	4.426	12.2	4.162	4.418	12.7
5.019	4.423	10.3	4.160	4.416	10.9
5.015	4.420	9.1	4.158	4.413	9.5
5.012	4.417	8.0	4.156	4.411	8.1
5.010	4.415	6.9	4.155	4.410	7.1
5.006	4.412	6.1	4.153	4.408	6.2
5.004	4.410	5.3	4.151	4.406	5.5
5.002	4.408	4.7	4.150	4.405	4.8
5.000	4.407	4.2	4.148	4.404	4.2
5.000	4.407	3.7	4.147	4.402	3.7
			4.147	4.402	3.2

In order to afford a means of comparison for the three curves, the three series of observations were all reduced to the same period, that of experiment *C*. To do this, the three curves were produced until they cut the *y*-axis and from these points of intersection, a reduction factor was obtained. All the values for the periods of the curve to be reduced were then multiplied by this factor. This operation will have the effect of making the two curves to be compared start with the same period at

zero amplitude. In Table IX. are shown the results for experiments *A* and *B* reduced in the manner described above so as to correspond to that of *C*.

These reduced values were plotted and the curves are shown in Fig. 7. Curve *A* represents the results for experiment *A* and curve *C* the results for experiments *B* and *C*. It is found from Fig. 7, that the curve for

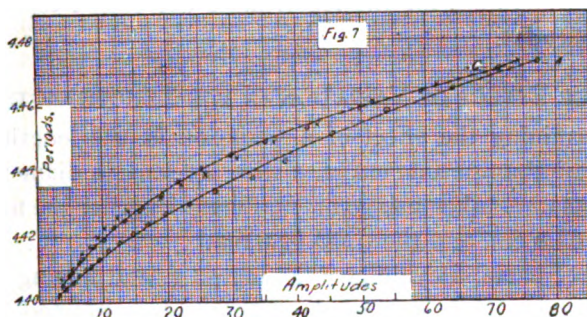


Fig. 7.

experiment *A* starts and finishes at about the same places as that for experiment *C*, but the middle portion lies quite a little below that for *C*. In other words, the curve for *A* is much flatter than that for *C*. On the other hand, the curves for *B* and *C* very nearly coincide. In fact, the variation in position of the two sets of points in the combined curve is no greater than the variation in the position of the points in either curve considered alone. When we remember that in observations *A* and *C*, both the mass of the vibrating system and its moment of inertia were varied, while in *B* and *C* only the moment of inertia was varied, we must conclude that, if we keep the stress to which the wire is subjected constant, the period will be the same function of the amplitude no matter what the moment of inertia, while if the stress as well as the moment of inertia is varied, the period will be a different function of the amplitude.

From the formula for the period, $T = 2\pi \sqrt{I/T}$, we should expect the reduction factor in each case to be equal to the square root of the ratio of the corresponding moments of inertia. The reduction factor for reducing observation *A* to observation *C* is 1.061. The value of $\sqrt{I_A/I_B}$ is 1.059. The reduction factor for reducing *C* to *B* is .881 while the value of $\sqrt{I_B/I_C} = .879$. When we remember that the factors here given were obtained from the curves, the agreement is remarkably close.

For lack of space the curves showing the relation between the logarithmic decrements and amplitudes in this experiment are not shown but it was found that here again we have a very good agreement between the curves for experiments *B* and *C* while the curve for experiment *A* differs to quite a marked extent from the other two, the curve for experiment *A*

lying below that for experiments *B* and *C*. This difference is especially noticeable in that portion of the curve corresponding to the larger amplitudes. This indicates that the shape of the logarithmic decrement-amplitude curve is also independent of the moment of inertia of the suspended weight but does depend upon the stress to which the wire is subjected. The effect of increasing the mass of the vibrating system is to increase the values for the logarithmic decrement especially for the higher amplitudes.

VI. THE EFFECT OF VARIATION IN THE LENGTH OF THE WIRE.

For determining the effect of a variation in the length of the wire upon the period-amplitude curve, a wire .25 mm. in diameter was used. Observations were taken using three different lengths: the first 104.8 cm., the second 84.0 cm. and the third 62.6 cm.

In all three cases, a cylinder weighing 50 g. and having a moment of inertia of 133 g. cm^2 . was used. The results obtained are given in Table X.

TABLE X.

Diam., .25 mm. Moment of inertia, 133 g. cm^2 .

No. of Swing.	Exp. A. Length 104.8 cm. Initial Amp. 175°.4. Temp. 26°.0.		Exp. B. Length 84.0 cm. Initial Amp. 136°.9. Temp. 26°.5.		Exp. C. Length 62.6 cm. Initial Amp. 105°.3. Temp. 27°.0.	
	Period.	Amplitude.	Period.	Amplitude.	Period.	Amplitude.
5	5.306	149.1	4.734	116.1	4.131	88.8
10	5.296	121.5	4.725	94.5	4.126	72.3
15	5.290	100.0	4.717	77.2	4.120	59.6
20	5.283	82.6	4.710	63.7	4.115	49.2
25	5.277	68.7	4.704	52.7	4.109	40.8
30	5.271	57.2	4.699	43.9	4.104	34.1
35	5.265	47.9	4.695	36.6	4.100	28.5
40	5.260	40.3	4.690	30.7	4.097	24.0
45	5.256	33.9	4.686	25.8	4.094	20.2
50	5.252	28.9	4.682	21.9	4.091	17.3
55	5.248	24.5	4.679	18.5	4.089	14.7
60	5.245	21.0	4.675	15.8	4.085	12.6
65	5.242	17.8	4.673	13.6	4.082	10.8
70	5.239	15.4	4.670	11.6	4.080	9.2
75	5.237	13.3	4.668	9.9	4.078	8.1
80	5.235	11.5	4.667	8.6	4.076	7.0
85	5.233	9.9	4.666	7.4	4.075	6.1
90	5.232	8.6	4.664	6.5	4.073	5.3
95	5.231	7.5	4.663	5.6	4.072	4.5
100	5.230	6.5	4.662	4.9	4.070	4.0
105	5.230	5.6	4.661	4.3	4.069	3.5
110	5.229	5.0			4.068	3.1
115	5.228	4.4			4.067	2.7
120	5.227	3.8				

In this case, as in the determination of the effect of a change in moment of inertia, it is necessary to reduce all experiments to the same period in order to obtain a basis for comparison. Experiment *B* was chosen as the basis for the comparison and the periods in the other two series reduced to this. The factors for reduction were obtained in the same way as before, that is, by determining the points of intersection of the curves with the *y*-axis. In this way, the curves will all start with the same period at zero amplitude. The reduced values for experiments *A* and *C* are given in Table XI.

TABLE XI.

A factor = .8912.*C* factor = 1.1458.

Old Period.	New Period.	Amplitude.	Old Period.	New Period.	Amplitude.
5.306	4.729	149.1	4.131	4.733	88.8
5.296	4.720	121.5	4.126	4.727	72.5
5.290	4.715	100.0	4.120	4.720	59.6
5.283	4.708	82.6	4.115	4.715	49.2
5.277	4.703	68.7	4.109	4.708	40.8
5.271	4.698	57.2	4.104	4.702	34.1
5.265	4.692	47.9	4.100	4.698	28.5
5.260	4.688	40.3	4.097	4.694	24.0
5.256	4.684	33.9	4.094	4.691	20.2
5.252	4.680	28.9	4.091	4.687	17.3
5.248	4.677	24.5	4.089	4.684	14.7
5.245	4.674	21.0	4.085	4.680	12.6
5.242	4.672	17.8	4.082	4.677	10.8
5.239	4.669	15.4	4.080	4.675	9.2
5.237	4.667	13.3	4.078	4.672	8.1
5.235	4.665	11.5	4.076	4.670	7.0
5.233	4.664	9.9	4.075	4.669	6.1
5.232	4.663	8.6	4.073	4.667	5.3
5.231	4.662	7.5	4.072	4.665	4.5
5.230	4.661	6.5	4.070	4.663	4.0
5.230	4.661	5.6	4.069	4.662	3.5
5.229	4.660	5.0	4.068	4.661	3.1
5.228	4.659	4.4	4.067	4.660	2.7
5.227	4.658	3.8			

In reducing the results of experiments *A* and *C* to compare with those of *B*, we should expect the factor in each case to be equal to the ratio of the square roots of the lengths, since the period of vibration varies as the square root of the length. The factor for the reduction of the data of experiment *A* to those of experiment *B* is .8912. The value $\sqrt{L_B/L_A}$ is .8846, giving a difference of .0066. The factor for the reduction of *C* to *B* is 1.1458. The value of $\sqrt{L_B/L_C}$ is 1.1445, giving a difference of .0013.

For lack of space the curves for these results are not shown but it was found that the curves did not coincide in this case as they did in the moment of inertia experiment. The curves for the shorter lengths were found to be much steeper than for the longer length. The amounts of decrease in period were about the same in all three cases. It must be remembered, however, that in this experiment we are not comparing corresponding points in the wire. That is to say, the point whose arc of vibration we are measuring in experiment *C* would vibrate through a much larger arc for the same amount of twist in the wire, if the point were at the end of a longer wire such as we have in experiment *A*. For this reason, a reduction factor must be applied to the amplitudes as well as to the period. These reduction factors were obtained by comparing the amplitudes corresponding to the same period in the two curves under consideration. All the amplitudes in the given experiment were then multiplied by this reduction factor. In this reduction, experiment *B* was taken as the standard and the amplitudes of *A* and *C* reduced so as to correspond. The reduced results for experiments *A* and *C* are shown in Table XII.

TABLE XII.

Experiment A. Factor = .732.

Experiment C. Factor = —.

Old Amp.	New Amp.	Period.	Old. Amp.	New Amp.	Period.
149.1	110.1	4.729	88.8	128.7	4.733
121.5	88.9	4.720	72.5	105.1	4.727
100.0	73.2	4.715	59.6	86.4	4.720
82.6	60.5	4.708	49.2	71.1	4.715
68.7	50.3	4.703	40.8	59.2	4.708
57.2	42.4	4.698	34.1	49.4	4.702
47.9	35.1	4.692	28.5	41.3	4.698
40.3	29.5	4.688	24.0	34.8	4.694
33.9	24.8	4.684	20.2	29.3	4.691
28.9	21.1	4.680	17.3	25.0	4.687
24.5	17.9	4.677	14.7	21.3	4.684
21.0	15.4	4.674	12.6	18.3	4.680
17.8	13.0	4.672	10.8	15.7	4.677
15.4	11.3	4.669	9.2	13.3	4.675
13.3	9.6	4.667	8.1	11.7	4.672
11.5	8.1	4.665	7.0	10.2	4.670
9.9	7.2	4.664	6.1	8.8	4.669
8.6	6.3	4.663	5.3	7.7	4.667
7.5	5.5	4.662	4.5	6.5	4.665
6.5	4.7	4.661	4.0	5.8	4.663
5.6	4.1	4.661	3.5	5.1	4.662
5.0	3.7	4.660	3.1	4.5	4.661
4.4	3.2	4.659	2.7	3.9	4.660
3.8	2.8	4.658			

The curves obtained from these results together with the original curve for experiment *B* are shown in Fig. 8. We see from that figure that we have a remarkably close agreement between the three sets of points.

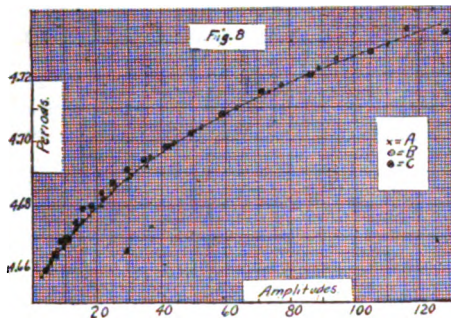


Fig. 8.

If the curves for these experiments were straight lines, we would expect the reduction factors for the amplitudes to be equal to the ratio of the lengths. As it is they are quite different. The reduction factor for changing *A* to *B* is .732 while the ratio of the lengths is .782. The reduction factor for the amplitudes of *C* is 1.450, while the ratio of the lengths is 1.310.

We must conclude from the fact that by merely applying constant factors to the two variables in these observations, the three curves can be made to coincide, that the period in each case is the same function of the amplitude. It also indicates that the wire is uniform throughout its length and that its peculiar behavior is not due to any particular portion of the wire.

VII. THE EFFECT OF VARIATION IN DIAMETER ON THE PERIOD-AMPLITUDE CURVE AND THE LOGARITHMIC DECREMENT-AMPLITUDE CURVE.

To determine the effect of varying the diameter of the wires upon the period-amplitude curve and the logarithmic decrement-amplitude curve, two wires of different diameter were used with suspended weights of such a mass that the stress to which the wire was subjected was the same in both cases.

In this experiment, the wires were of .25 mm. and .5 mm. in diameter respectively. Since the cross-sectional area of the first was one fourth that of the second, the mass of the suspended weight in the first case was one fourth that in the second. Masses of 50 g. and 200 g. respectively were used.

TABLE XIII.

No. of Swing.	Exp. A. Diam. .35 mm. Length 97.9 cm. Mass 50 g. Moment of Inertia 133 g. cm ² . Temp. 28° C.		Exp. C. Diam. .5 mm. Length 97.9 cm. Mass 200 g. Moment of Inertia 1,456 g. cm ² . Temp. 26° C.	
	Period.	Amplitude.	Period.	Amplitude.
5	5.320	142.4	4.521	72.9
10	5.309	114.7	4.514	60.7
15	5.299	92.7	4.509	50.7
20	5.292	75.6	4.503	42.8
25	5.285	62.1	4.499	35.9
30	5.279	51.2	4.496	30.5
35	5.274	42.4	4.491	25.8
40	5.268	35.3	4.487	22.2
45	5.263	29.6	4.483	18.8
50	5.259	24.9	4.478	16.1
55	5.256	21.2	4.475	13.8
60	5.253	17.9	4.472	12.1
65	5.251	15.4	4.468	10.4
70	5.249	13.1	4.466	9.1
75	5.246	11.3	4.464	8.0
80	5.243	9.3	4.462	7.0
85	5.241	8.4	4.459	6.0
90	5.239	7.2	4.458	5.3
95	5.238	6.2	4.457	4.6
100	5.237	5.4	4.456	4.0
105	5.236	4.8		
110	5.235	4.1		
115	5.235	3.6		
120	5.234	3.2		

In this experiment, it was necessary to use a smaller initial amplitude for the wire of large diameter than for the one of small diameter because of the lower elastic limit in the one case than in the other. Of course,

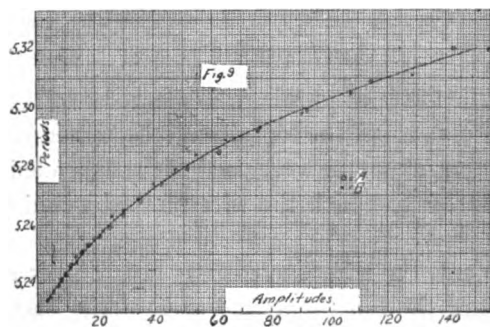


Fig. 9.

here again, we must apply a reduction factor to the periods in order to afford a basis of comparison. This was done, the factor being obtained

in the same way as before. Experiment *A* was taken as the standard and *B* reduced to correspond.

The factor for reducing the period should theoretically be equal to the inverse ratio of the squares of the diameters multiplied by the ratio of the square-roots of the moments of inertia. This operation gives a factor of 1.208, while that found from a comparison of the curves was 1.176.

If a second reduction factor be applied to the amplitudes it is found that the curves coincide. These reduced results are shown in Table XIV. and the two sets of points are plotted in Fig. 9.

TABLE XIV.

Factor = 2.12.

Old Amp.	New Amp.	Period.	Old Amp.	New Amp.	Period.
72.9	154.5	5.320	13.8	29.2	5.265
60.7	128.7	5.311	12.1	25.7	5.263
50.7	107.5	5.305	10.4	22.5	5.258
42.8	90.7	5.298	9.1	19.3	5.255
35.9	76.1	5.293	8.0	17.0	5.253
30.5	64.7	5.289	7.0	14.9	5.250
25.8	54.7	5.284	6.0	13.2	5.247
22.2	47.1	5.279	5.3	11.2	5.245
18.8	39.9	5.274	4.6	9.8	5.243
16.1	34.1	5.269	4.0	8.0	5.241

From the fact that the curves for the two observations can be made to coincide by applying constants to the two variables, we must conclude that when the stress to which the wire is subjected is kept constant, the period is the same general function of the amplitudes no matter what the diameter of the wire.

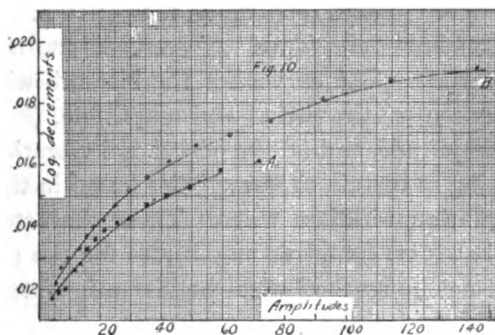


Fig. 10.

The Logarithmic Decrement-Amplitude Curve.—The logarithmic decrements in these two experiments were computed and plotted against the corresponding values of the amplitudes. The results of these computations are shown in Table XV. and the curves in Fig. 10.

TABLE XV.

Exp. A.			Exp. B.		
No. of Swings.	Log. Dec.	Amplitude.	No. of Swings.	Log. Dec.	Amplitude.
8	.0191	142.2	8	.0161	71.3
10	.0186	114.7	10	.0158	59.4
10	.0181	92.7	10	.0153	49.6
10	.0174	75.6	10	.0150	41.8
10	.0169	62.1	10	.0147	35.1
10	.0166	51.2	20	.0143	29.8
10	.0161	42.4	20	.0141	25.2
10	.0156	35.3	20	.0139	21.7
10	.0152	29.6	20	.0136	18.4
20	.0147	24.9	20	.0133	15.7
20	.0142	21.2	20	.0128	13.5
20	.0140	17.9	40	.0126	11.8
20	.0137	15.4	40	.0120	8.9
20	.0133	13.1	40	.0118	6.8
20	.0130	9.7			
20	.0127	7.2			
20	.0122	5.4			
20	.0117	4.1			

It will be noticed from Fig. 10 that while the logarithmic decrement curves start from about the same point, the one for the small diameter is steeper than that for the wire of larger diameter.

SUMMARY.

1. The elastic properties of bismuth wires have been investigated and it has been found that they display some of the same peculiar properties shown by the platinum-iridium wires investigated by Guthe and Sieg. When the wires were vibrated torsionally, the period of vibration and the logarithmic decrement were found to decrease enormously with the decrease in amplitude. The curves for the bismuth wires representing the relations between (a) the period and amplitude, (b) the logarithmic decrement and amplitude, and (c) the period and vibration number, were found to be lacking in certain characteristics found in the corresponding curves for the platinum-iridium wires. These characteristics were for the most part, however, found at amplitudes for the platinum-iridium wires that could not be reached with the bismuth wires, because of the low elastic limit of the latter.

2. It was found that, in two experiments in which the mass of the vibrating system was kept constant but the moment of inertia varied, the period-amplitude curves in the two cases could be made to coincide by applying a reduction factor to the periods in one set of observations.

This reduction factor was found to be equal to the ratio of the square roots of the moments of inertia used in the two experiments. This indicates that the mathematical relation between the period and amplitude does not depend upon the rapidity of motion of the vibrating system. The logarithmic decrement-amplitude curves in the two cases were found to coincide.

If the load as well as the moment of inertia was varied, it was found that when the proper reduction factor was applied to the periods, the two curves did not coincide, the curve corresponding to the vibrating system having the smaller mass being flatter than the other. The curves for the logarithmic decrement did not coincide, the greater values for the logarithmic decrement being found in the case in which the mass of the suspended weight was greatest.

3. When the length of the wire was varied, it was found that by applying reduction factors to the periods and amplitudes, the period-amplitude curves for the different experiments could be made to coincide, indicating that the period is the same general function of the amplitude in all cases. This also indicates that the wire is uniform throughout its length and that its peculiar behavior is not due to any particular section of the wire.

4. Two wires of different diameter were used with suspended weights of such a mass that the stresses to which the wires were subjected were the same in each case. In this experiment, also, by applying the proper factors to the amplitudes and periods the curves could be made to coincide. The logarithmic decrement curve for the smaller wire was found to lie above that for the wire of larger diameter, indicating a greater damping effect in the former.

In conclusion, the writer wishes to express his appreciation of the helpful advice and kindly interest of Prof. K. E. Guthe, at whose suggestion this investigation was undertaken.

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THE HALF-VALUE OF THE RADIOACTIVE DEPOSIT COLLECTED IN THE OPEN AIR.

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IT has been quite definitely settled by experiment that the radioactive deposit collected in the open air on a negatively charged wire consists of a mixture of the disintegration products of radium and thorium.¹ The proportion of the two active deposits varies, depending upon the number of hours of exposure, potential of the wire, the atmospheric conditions, as well as the locality. The question of the half-value however seems to be still an open one. S. J. Allen and Harvey find that the half-value is not a constant but varies between wide limits. Other observers have maintained that the half-value is constant or varies but slightly.

It is the object of the present paper to show, from theoretical considerations, that the half-value should vary; to show how it should vary, and to compare theory with experiment.

When a negatively charged body is exposed to thorium emanation the manner in which the active deposit collected on the wire decays depends on the number of hours of the exposure. If the exposure is short, *i. e.*, less than about 6 hours, the number of radioactive particles transforming per second increases during the first few hours (3 to 6 hours) then gradually decreases and finally decays regularly according to an exponential law. After this regular decay in activity begins, half-value is reached in about eleven hours. Even when the exposure is of 18 to 24 hours duration the activity does not begin at once to decay logarithmically but increases very slightly for about an hour, then begins gradually to decrease, and after three or four hours decays logarithmically with half-value 10.6 hours, until the activity disappears. The activity, at any time subsequent to the removal from the emanation of the wire on which the active deposit is collected, in terms of the initial activity, may be calculated from the equation

¹ H. A. Bumstead, *Amer. Jour. Sc.*, July, 1904; H. M. Dadourian, *Amer. Jour. Sc.*, Jan., 1905; S. J. Allen, *Phil. Mag.*, Dec., 1904, and *Phys. Rev.*, June, 1908; G. A. Blanc, *Phil. Mag.*, Mar., 1907; D. Pacini, *Phys. Zeit.*, Mar. 15, 1910; F. A. Harvey, *Phys. Rev.*, Mar., 1909.

$$I = I_0 \frac{ae^{-\lambda_2 t} - be^{-\lambda_1 t}}{a - b},^1$$

where t is the time, in seconds, from the instant of discharge of the wire, *i. e.*, from the instant when it ceases to collect active material from the transforming emanation, $\lambda_1 = .000208$, $\lambda_2 = .0000182$ and a and b are constants depending on the time of exposure T , to the emanation, in seconds.

$$a = \frac{1 - e^{-\lambda_2 T}}{\lambda_2}, \quad b = \frac{1 - e^{-\lambda_1 T}}{\lambda_1}.$$

This theory has been tested experimentally by Miss Brooks,² Her results show good agreement between theory and experiment, except for exposures less than one hour. Miss Brooks obtained her results with exposures of 1, 2, 3, 4 and 6 hours. In comparing experimental with theoretical values she used the constants $\lambda_1 = .00021$ and $\lambda_2 = .0000175$. These values have since been superseded by the more accurate ones given above.

From the equation above, I have calculated the values which the activity on the wire would have at different times up to 50 hours after removal from the emanation. In order that the comparison with open air exposures may be made, calculations are for 3, 4, 12 and 22 hour exposures, these being frequent times used in open air experiments.

The results of this calculation are given in Table I. Figure 1 shows

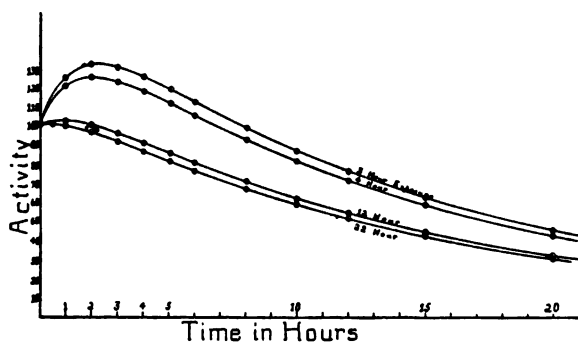


Fig. 1.

Curves showing rise and decay of the active deposit from thorium emanation for different lengths of exposure.

the same thing graphically. It is to be noted that the activity at first increases in intensity. Even for the 22 hour exposure the curve does

¹ Rutherford, Radioactivity, page 351 et sequa.

² Phil. Mag., Vol. 8, 1904, p. 373.

not become logarithmic until about $3\frac{1}{2}$ to 4 hours. For this long exposure the activity increases for the first half hour.

TABLE I.

Time after Activation.	3 Hour Exposure.	4 Hour Exposure.	12 Hour Exposure.	22 Hour Exposure.
0 hours	100	100	100	100
$\frac{1}{2}$ hour				100.4
1 hour	124.8	120.5	102.4	99.8
2 hours	131.7	124.8	100.3	96.3
3 hours	130.3	122.7	96.0	91.6
4 hours	125.2	117.6	90.8	86.4
5 hours	118.8	111.4	85.6	81.3
6 hours	112.0	105.1	80.3	76.3
8 hours	98.8	92.6	70.9	67.0
10 hours	86.8	81.5	62.0	58.8
12 hours	76.3	71.5	54.4	51.6
15 hours	62.7	58.8	44.7	42.4
20 hours	45.3	42.3	32.2	30.5
30 hours	24.3	22.0	16.4	15.9
50 hours	6.3	5.9	4.5	4.3

The curve of decay for the active deposit from radium might also be calculated, but, owing to the fact that there is a larger number of transformation products, the calculation would be much more complicated. For this reason an experimental curve has been used. This experimental curve has been compounded with the curve shown in Fig. 1 for the four hour exposure and the result is shown in Table II. Table II. shows, then, for a four hour exposure, the decay of the activity on a wire where this activity consists of a mixture of radium and thorium disintegration products, in varying proportions. The activity is taken as 100 at the ten minute point, *i. e.*, ten minutes after the wire is removed from the emanation, as in open air experiments it takes about ten minutes before observations can be commenced. Also the radium A has practically all transformed by this time and the initial activity is thus much easier to determine. The abbreviations "Th" and "Ra" are used to indicate the disintegration products of thorium and radium respectively. Any column in Table II. will give the activity of a mixture of thorium and radium active deposits in the percentages indicated at the head of the column for the times given in the first column.

These values are plotted in Fig. 2 (circle points). From these curves for the decay of the mixture of active products it is a simple matter to determine the half-value period. The curve drawn in Fig. 2 with X's shows the variation of the half-period. The variation is at first slight, *i. e.*, until about 30 per cent. is reached, then it begins to suddenly become

TABLE II.

Four Hour Exposure to a Mixture of Emanations.

Time.	Th, 100 Per Cent., Ra, 0 Per Cent.	Th, 0 Per Cent., Ra, 100 Per Cent.	Th, 10 Per Cent., Ra, 90 Per Cent.	Th, 20 Per Cent., Ra, 80 Per Cent.	Th, 30 Per Cent., Ra, 70 Per Cent.	Th, 40 Per Cent., Ra, 60 Per Cent.	Th, 50 Per Cent., Ra, 50 Per Cent.	Th, 60 Per Cent., Ra, 40 Per Cent.	Th, 70 Per Cent., Ra, 30 Per Cent.	Th, 80 Per Cent., Ra, 20 Per Cent.
0	100	100	100	100	100	100	100	100	100	100
10 min.	100	100	100	100	100	100	100	100	100	100
20	104.3	87.4	98.1	90.8	92.5	94.3	96.0	97.7	99.4	101.1
40	110.8	68.5	72.8	77.0	81.1	85.4	89.6	93.9	98.1	102.4
60	115.8	48.9	55.6	62.3	69.0	75.6	82.3	88.1	95.8	102.5
80	118.2	33.5	41.8	50.4	58.9	67.4	75.9	84.3	92.8	101.3
100	119.6	22.3	32.1	41.7	51.5	61.2	70.9	80.7	90.4	100.2
120	119.8	14.6	25.1	35.7	46.2	56.7	67.2	77.7	88.2	98.7
3 hrs.	117.9	3.7	15.1	26.6	38.0	49.4	60.7	72.2	83.6	95.0
4	113.0	.6	11.8	23.1	34.3	45.6	56.5	68.0	79.4	90.5
5	107.0	.18	10.9	21.5	32.2	42.9	53.6	64.2	74.9	85.7
6	101.1	.16	10.3	20.3	30.4	40.5	50.6	60.7	70.8	80.9
8	89.1	0	8.9	17.8	26.7	35.6	44.5	53.4	62.3	71.3
10	78.3	0	7.8	15.7	23.5	31.3	39.1	47.0	54.8	62.7
12	68.7	0	6.9	13.7	20.6	27.5	34.3	41.2	48.0	54.8
15	56.6	0	5.7	11.3	17.0	22.6	28.3	33.9	39.6	45.2
18	46.2	0	4.6	9.2	13.8	18.5	23.1	27.7	32.3	36.9
20	40.7	0	4.1	8.1	12.2	16.3	20.4	24.4	28.5	32.5
30	21.1	0	2.1	4.2	6.3	8.4	10.5	12.7	14.8	16.9
50	5.8	0	.6	1.2	1.7	2.3	2.9	3.5	4.1	4.6

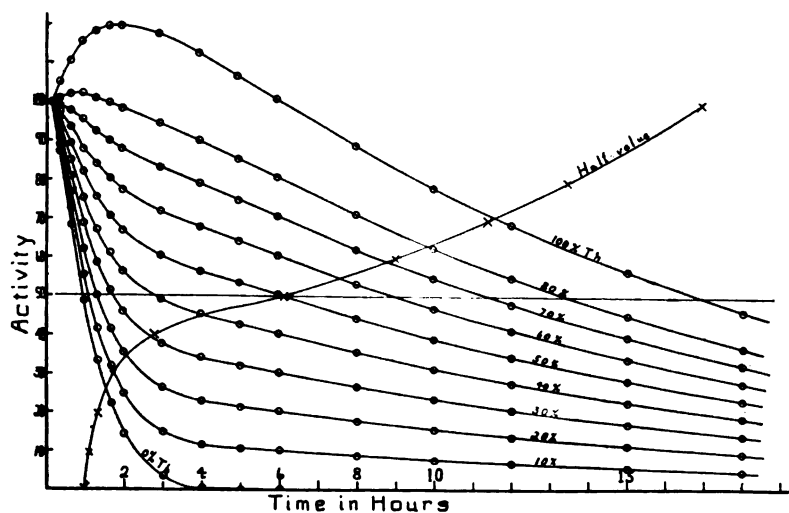


Fig. 2.

Theoretical curves for a mixture of radium and thorium active deposits, in varying proportions, for a four hour exposure. Half-value curve. The ordinates for the half-value curve are in per cent. with the same magnitude as the scale for the activity.

much greater and increases rapidly as the percentage of thorium active deposit increases until the limiting value, about $16\frac{1}{2}$ hours is reached. From 20 per cent. to 40 per cent. the half-value increases from a little over 60 minutes to nearly three hours. Note that in Fig. 2 since the initial activity is taken as 100 at 10 minutes the time for the half-value is always ten minutes too large. The values corresponding to Table II. for the three hour and twelve hour exposures have been calculated and the corresponding curves plotted. From these curves the time to sink to half-value has been taken and these values are given in Table III. The same results are shown graphically in Fig. 3. The three hour and

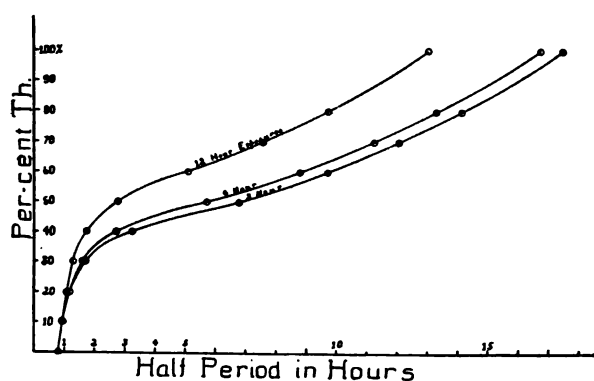


Fig. 3.

Curves showing variation of half-value of a mixture of active deposits with the percentage of thorium active deposit.

twelve hour exposures show the same general characteristics as the four hour exposure. In the three hour exposure there is a greater variation of the half-period and in the twelve hour it is slightly less. It is evident

TABLE III.

Per Cent. Th Active Deposit.	Time to Sink to Half-value.		
	3 Hr. Exposure.	4 Hr. Exposure.	12 Hr. Exposure.
0	47 min.	47 min.	47 min.
10	58	57	55
20	71	70	64
30	100	95	77
40	194	162	104
50	7 hr. 46 min.	5 hr. 42 min.	2 hr. 45 min.
60	9 hr. 40 min.	8 hr. 46 min.	5 hr. 5 min.
70	12 hr. 1 min.	11 hr. 12 min.	7 hr. 33 min.
80	14 hr. 5 min.	13 hr. 15 min.	9 hr. 41 min.
100	17 hr. 25 min.	16 hr. 41 min.	13 hr. 0 min.

that if the percentage of the active deposit from thorium varies up to 60 per cent. and 70 per cent., as experimental results have indicated it does, that the half-value must vary between very wide limits. In experiments performed by the writer active deposits containing a large percentage of thorium transformation products have been obtained and other observers, in particular Blanc,¹ have obtained even higher percentages.

TABLE IV.

No. Curve.	Hours Exposure.	Per Cent. Th.	Half-value.		Barometer.
			Observed.	Calculated.	
1	3	1 or 2	48 min.	45 min.	Rising rapidly.
2	3	1 or 2	48	43	Falling.
3	3	2	49	52	Rising rapidly.
4	3	3	51	53	Rising.
5	3	3.5	52	51.5	Steady.
6	3	5	53	45	Steady.
7	3	6	54	50	Falling slowly.
8	3	9.5	58	59	Steady.
9	3	10	58	53	Steady.
10	3	20	71	71	Steady.
11	4	1 or 2	48	47	Steady.
12	4	9	56	52	Nearly steady.
13	4	10	57	57	Steady.
14	4	19	69	50	Steady.
15	12	8	54	46	Wind stopped.
16	12	29	77	78	Rising.
17	13	10	54	52	Steady.
18	13	12	56	50	Falling.
19	13	62	5 hr. 45 min.	2 hr. 30 min.	Falling.
20	14	34	86	85	Steady.

It cannot be expected that there will be perfect agreement between theory and experiment in an exposure made in the open air. Calculations can only be made on a basis of a constant supply of emanations and in an open air exposure this is not likely to be realized. If the barometer changes rapidly just before or during an experiment the supply of thorium emanation will be changed, as has been shown by the writer.² If the wind changes in direction or speed it is probable that there will be a change in the ratio of the amounts of emanation in the air. Considering these facts, the agreement is surprisingly good. Table IV. shows the results of 20 experiments. Of these 20 experiments seven show only a slight difference from the calculated value; a difference of only

¹ Phil. Mag., 13, p. 378, March, 1907.

² Loc. cit.

one minute. This is within the error of calculation as the graphical method used will only give half values within about one minute. Thirteen results are within four minutes. When the agreement is not good it is nearly always plainly traceable to barometric change as may be seen from Table IV. There is only one experiment where the barometer is steady where the agreement is not at least fair and in that case there was a sudden decrease in the speed of the wind just before the start of the experiment.

The experiments by D. Pacini¹ do not show quite so good agreement with the above theory. Out of 22 experiments the barometer was steady or changed slightly in ten. None of the observations during which the barometer changed agree with the theory. This is to be expected as was pointed out above. Of the ten others, five show good agreement and one experiment where the barometer was steady was accompanied by a sudden change in the wind. There is no apparent reason why the other four results should not agree with the theory. Table V. shows a comparison between the half period found by Pacini and that calculated from the above theory for the nine experiments where agreement is to be expected. The percentages of thorium active deposit are of course the actual percentages on the wire and not the equilibrium values.

TABLE V.

No. Curve.	Hours Exposure.	Per Cent. Th.	Half-value.		Barometer.
			Observed.	Calculated.	
2	4	16.5	65	65	Slow rise.
11	4	15.5	61	64	Slow rise.
12	3½	12.1	59	59	Slow rise.
13	4	12.5	56	60	Slow rise.
14	4	8.5	52	55	Slow rise.
15	4	26.8	61	83	Slow rise.
18	4	12.6	53	60	Slight rise.
19	4	15.9	57	64	Slight rise.
20	4	11.7	56	58	Slight rise.

SUMMARY.

It is shown from theoretical considerations that the half-value of a mixture of the active deposits from radium and thorium emanations must vary between very wide limits (45 minutes to several hours) if the percentage of thorium active deposit varies from 0 per cent. to 50-70 per cent. It is further shown that when atmospheric conditions are

¹ Phys. Zeit., March 15, 1910, p. 209.

favorable, *i. e.*, steady barometer and only slight changes in the wind, very good agreement is obtained between theory and experiment for the mixture of active deposits collected on a negatively charged wire exposed in the open air.

STELLE HALL OF PHYSICS,
SYRACUSE UNIVERSITY,
May 10, 1912.

THE MELTING OF ICE IN A MIXTURE OF ICE AND WATER.

BY E. E. SOMERMEIER.

THE results given in this paper were obtained by the author in connection with some calorimetric work in which a two-liter Dewar vacuum flask was used as a calorimeter. In testing out the possibilities of the flask as a calorimeter the determination of the latent heat of fusion of ice suggested itself and a series of determinations were run as follows:

1,600 grams of water were accurately weighed in the Dewar flask. A thermometer graduated in .01 of a degree and readable by means of a telescope to .001 of a degree was inserted through a cork into the flask and temperature readings were taken from time to time, the contents of the flask being well mixed by shaking the flask between readings. After a series of readings had been obtained a weighed amount of ice was poured into the calorimeter and the latent heat of fusion determined by calculations based upon temperature changes and the amounts of water and ice used. Details of the calculations, data on the water equivalent of the calorimeter, thermometer corrections, etc., are omitted here as not being directly related to the subject matter of this article.

The method of work in so far as ice is concerned was as follows: Sticks of ice about one centimeter in diameter were frozen in glass tubes and after removal from the tubes the sticks were allowed to freeze dry in a room the temperature of which was several degrees below freezing. From 100 to 140 grams were then weighed in a cold Erlenmeyer flask. The flask was stoppered and immersed to the neck in a mixture of ice and water and allowed to remain for a considerable period of time. The mixture containing the flask and ice was then carried into the calorimeter room, the stopper of the flask removed, the neck grasped firmly by means of a handkerchief wrapped around it and the contents quickly poured into the calorimeter, the operation requiring only the fraction of a minute. The Erlenmeyer flask was then stoppered and set to one side while the temperature readings on the thermometer of the calorimeter flask were taken. The greater part of the ice quickly melted and within two minutes the contents of the calorimeter flask were within one to three degrees of the room temperature. With this temperature difference the temperature change per minute was only .0001 to .0003° and readings were taken

only at intervals of four or five minutes until a sufficient number had been taken to fix an end point and to establish a final rate, from which the temperature changes and radiation corrections were determined.

The Erlenmeyer flask in which the ice was weighed was wiped thoroughly dry on the outside and placed in the cold room and allowed to stand until at room temperature. It was then weighed and the difference between this weight and the weight with the ice sticks, taken as the weight of ice added to the calorimeter.

In the first series of determinations made in this way the amounts of ice used in a determination varied from 99 to 140 grams and the sticks were weighed at minus 7° and allowed to stand in the mixture of ice and water from 45 to 60 minutes. The idea of the writer was that the sticks, which were below the freezing temperature when put into the mixture, would in time attain practically the zero temperature of the mixture around the flask. The results obtained for the latent heat of fusion in this series of determinations are as follows: 79.88, 79.98 and 79.89 or an average of 79.92. These results were fairly satisfactory, but it seemed desirable to have an average of a larger number of determinations and a second series was begun.

In considering the results of the first determinations the only error that might be eliminated which suggested itself was the possibility that the sticks had not been allowed to stand in the mixture of ice and water sufficiently long to have arrived at approximately zero temperature. Accordingly in the second series, the sticks were immersed for a much longer period, the time varying from 70 to 200 minutes. The amounts of ice taken in this series of determinations varied as in the first series from about 100 grams up to about 140 grams. The second series of results checked fairly well among themselves, being as follows: 79.53, 79.56, 79.38, and 79.51, an average of 79.49. However, the average of this series does not agree at all with the average of the first series, being .4 lower. The greatest difference in conditions between the two series was that in the first the ice was weighed at minus 7° and immersed from 45 to 60 minutes, in the second it was weighed at minus $1\frac{1}{2}^{\circ}$ to minus 3° and allowed to set from 70 to 200 minutes. Careful inspection of temperature rates, radiation corrections and points of manipulation in the two series showed no differences that would account for such a variation in results obtained and it seemed evident that the difference was in some way related to the difference in time of immersion in the mixture of ice and water. It was observed in the second series of results that the Erlenmeyer flasks after the ice was poured from them always contained a considerable amount of moisture, as much as a cubic centimeter

or so. Small amounts of moisture in the flasks might well have been due to breaking off of small particles from the cold brittle sticks when they were put into the flask and these small particles adhering to the sides of the flask upon melting would account for some moisture always being present, but the large amount present did not seem accountable for in this way. It did not seem possible that any of the ice had actually melted while immersed and that the disagreement in the calorimetric results was due to the fact that the sticks in the latter series when poured into the calorimeter were wet instead of dry. However in order to definitely determine this point tests were made as follows: Sticks of cold dry ice were suspended by fine wires in the Erlenmeyer flasks and the flasks immersed in ice and water as in the calorimeter experiments. Upon withdrawing the sticks drops of water were plainly visible upon the lower ends of the sticks and upon touching with filter paper appreciable amounts of water were absorbed by the paper.

Quantitative tests were then started in order to determine the amounts actually melted. In order to guard against errors due to any difference in water the same supply of distilled water recently boiled to expel air was used in freezing the sticks and in the freezing mixture. Details of treatment are as follows: The sticks of ice were handled by means of small wires looped around them and were allowed to remain in a cold porcelain evaporating dish or casserole until there was a certainty of their being frozen entirely dry if by any chance any melting should occur during the attachment of the wires. Stoppered weighing tubes were used for weighing the sticks, which as soon as weighed were transferred to long test tubes immersed in the mixtures of ice and water which were contained in glass cylinders. These test tubes were at the temperature of the room, which was below 0° C., so that both tubes and sticks when inserted into the mixture were colder than the mixture itself and would gradually warm up to the temperature of the surrounding mixture. During practically all of the tests the room temperature was considered below 0° C. and the mixture was continually freezing during the period of the tests. The sticks upon being removed from the tubes were touched lightly to cold filter paper to remove any drops of water hanging to them. They were then quickly placed in weighing tubes which were then closed and weighed and the loss observed and if the experiments were to be continued the sticks again transferred to the tubes in the mixture of ice and water.

Some of the results obtained are as follows:

TABLE I.

Test No.	Approx. Weight of Stick, Gms.	Room Temp. Beginning.	Room Temp. at End.	Time in Hours.	Volume in C.c. of Mixture of Ice and Water.	Ice Melted in Grams.	Ice Melted per Hour, Grams.
1a	87	-.8	-3.0	15½	1,500	2.17	.140
1b	85	-3.0	-1.0	24½		0.84	.034
1c	84	-1.0	+ .2	11		.57	.052
2a	35	-.8	-1.4	5	380	.46	.092
2b	35	-1.4	-3.0	11		.32	.031
2c	35	-1.4	-0.8	8½		.10	.012
2d	34	-0.6	-0.8	2¼	380	.03	.012
2e	34	-0.8	-1.0	12¾		.23	.018
2f	34	-1.0	-0.4	3		.15	.050
2g	34	-0.4	+0.2	7¾		.27	.035
3a	50	-2.0	+0.2	32½	1,550	1.00	.031
4a	21	-1.0	-1.4	4¾	110	.40	.085
4b	21	-1.4	-3.0	8¾		.07	.008
5	28	-0.8	-3.0	16½	1,500	.36	.022
6	66	-1.8	-1.0	35	1,500	2.46	.070

In tests Nos. 1, 2, 4, and 6 the test tubes in which the sticks were placed were immersed in the mixture of ice and water to the top of the tubes which were corked loosely to prevent interchange of air during the period of the tests. In tests Nos. 3 and 5 the sticks were placed in a short wide test tube which was securely stoppered with a rubber stopper which had passing through it a piece of narrow bore glass tubing. The test tube and stopper was completely immersed but the end of the glass tube extended above the mixture so that the pressure within the tube and around the sticks of ice was at atmospheric pressure. In test No. 5, two tubes were used, the stick of ice being placed in a smaller tube placed within the tube which was immersed in the freezing mixture. The stick, therefore, did not come into contact with the outside tube. The water formed during the test remained in the inner tube, no water being discernible on the sides of the outer tube, indicating little or no vaporization or condensation.

In none of the tests except in 1c, 2g, and 3a, was the temperature of the room as high as 0° C. and in these tests the final weighings were purposely delayed till the temperature was slightly above freezing in order to start the series of determinations given in Table II.

In none of the tests was the mixture of ice and water allowed to freeze solid but in all of them shells of ice were formed on the sides and bottom of the cylinder and in test 4*b* so much ice had frozen out that only about one fourth of the original amount of mixture remained liquid.

In all the tests the losses are of considerable magnitude and in some of them the heat required to melt the amount of ice melted if expressed in degrees on the amount of mixture used amounts to several tenths of a degree. For example, No. 4*a* with .4 gram of ice melted and 110 c.c. of mixture used equals 32 calories on 110 grams equal to .29 of a degree of temperature if this heat comes from the confining mixture. In 1*a* + 1*b* + 1*c* with 3.58 grams of ice melted and 1,500 c.c. of mixture, this amounts to .19 of a degree.

The tubes in which the sticks were immersed in all cases contained perceptible amounts of water and result No. 6 was in fact obtained by weighing the increase due to water rather than the loss in weight of the sticks. That heat was obtained from outside sources sufficient in amount to cause this melting is not possible as special precautions were taken to insure against such an occurrence. The mixtures used in all cases were in the room for a considerable time before using and where there seemed any possible doubt of the mixtures being at zero, they were tested by a thermometer. In no case did the temperature of the mixtures differ more than .002° from the zero point of the thermometer used. In most of the tests the freezing solution contained considerable ice crystals and in some of the tests the formation of minute ice crystals was observed upon the walls of the tube itself within a few minutes after its immersion. That the magnitude of the results obtained are different in the different tests is not surprising as differences in the size of stick and test tube used might well cause large differences in the rate of melting. Nos. 2*d*, 2*e*, 2*f*, 2*g*, and 2*h* were on the same stick as Nos. 2*a*, 2*b*, and 2*c*, but a new mixture was used, the first having frozen so nearly solid as to make a new one necessary.

In seeking an explanation, the first idea of the author was that possibly the freezing of the ice is an intermittent process and that the heat liberated by freezing and crystallization actually warms the temperature of the water to above zero. This heat is given up to any substance with which the water comes in contact and that the ice crystals themselves are not entirely melted is due to the fact that they are backed up by ice considerably below the zero temperature, hence the amounts of crystals redissolved does not equal the amount of the crystals formed and the freezing out progresses and with sufficient time the solution ultimately becomes solid. The crystals in the tube immersed in the solution, if once melted

by the absorption of any heat from the surrounding water, remain melted and the tube and stick constitute a sort of trap for absorbing heat. If, however, the melting is due to heat liberated by the freezing of ice crystals in other parts of the solution, then the melting of the sticks should cease when the mixture no longer is freezing, as would be the case when the mixture is kept in a room, the temperature of which is slightly above freezing. Table II. gives the results of a series run to test the accuracy of this hypothesis.

TABLE II.

Test No.	Approximate Weight of Stick, Gms.	Room Temp. Beginning.	Room Temp. at End.	Time in Hours.	Ice Melted in Grams.	Ice Melted per Hour in Grams.
1d	84	+2	+1.4	12½	.52	.047
2h	34	+2	+1.4	12¼	.51	.042
3b	50	+2	+1.4	12½	.36	.029

Comparison of these results with 1b and 1c, 2f and 2g and with 3a, where the losses per hour are .040, .040, and .031 gram, show practically the same rate of melting as that obtained when the mixture was actually freezing in the room the temperature of which was as low as minus 3°. Evidently there must be some other explanation than the one offered.

Apparently the melting cannot be due to traces of salts in the distilled water as the outer portion of the stick was frozen first and is pure ice. Any traces of salts if present are in the interior of the stick. As the melting of the stick is on the outside the portion melted is pure ice or if not absolutely pure of the same degree of purity as that freezing out in the freezing mixture surrounding it.

A possible explanation is that the different portions of the stick are either under tension or pressure. In either case the melting point is lower than zero. Tracing the progress of the formation of a stick similar to that used in these experiments the freezing begins at the outer portion of the solution and a continuous shell of ice surrounding and enclosing a liquid interior soon forms. As freezing progresses the shell becomes thicker and thicker until the entire mass finally freezes solid. The expansion accompanying the freezing of the liquid interior exerts sufficient pressure on the outer shell of ice to cause it to burst the confining tube or to lengthen out and extend beyond this tube. The strains consequent to this stretching of the outer shell and compression of the inner portion relieve themselves partially by fracturing and cracking of the stick but the adjustment is not perfect and the frozen stick consists of an outer zone under tension and an inner zone under compression. In either case the melting point of the stick is lower than zero and the stick melts when surrounded by a mixture at this temperature.

If the lower melting point is a result of strains a release or adjustment of a relatively large portion of them is to be expected when the stick is first warmed to approximately zero and actually begins to melt. Comparison of the results of 1*a*, 2*a*, and 4*a*, with later periods of the test, as 1*b*, 2*b*, etc., actually show a much greater rate during the first period. Also the results of 4*b* where the rate of loss is only one tenth as great as in 4*a* is in harmony with the actual conditions of the experiment as the solution surrounding the tube had frozen almost solid and was under pressure and contained any traces of salts which may have been present in the original solution. Traces of salts and pressure both lower the freezing temperature of the surrounding mixture and hence by keeping the ice in the immersed tube cooler diminishes the rate of melting or causes it to cease entirely.

If the lower melting point of the stick is a result of strains, then ice frozen under conditions where strains are not produced should remain frozen under the conditions which cause the sticks to melt. Experiments on free ice crystals obtained from the surface of distilled water freezing in an open dish and on icicles frozen from distilled water showed that both the crystals and the icicles melted when placed in tubes and the tubes immersed in the mixture of ice and water. Apparently these crystals and icicles should be at least as free from strains as the ice formed in the mixture surrounding the tubes and an explanation based on strains appears to be disproven.

A suggested explanation is that ice on first forming freezes in an unstable form which later changes into more stable crystals having a lower melting point. Buchanan¹ calls attention to the fact that lake ice during cold weather shows no crystalline figure but that on beginning to thaw a vertical prismatic structure develops. This loosening or separation of the closely crowded prisms of cold ice undoubtedly took place in the sticks used in the experiments when they were warmed up to zero. Possibly this readjustment is the reason for the lower melting point but if so it must involve some change in crystalline structure, as mere readjustment is simply an adjustment of strains, which explanation seems disproven by the melting of free ice crystals and icicles.

Sutherland² considers ice as $(\text{H}_2\text{O})_3$ and water as a mixture of $(\text{H}_2\text{O})_2$ and $(\text{H}_2\text{O})_3$ in varying proportions and states that the true physical melting point of ice $(\text{H}_2\text{O})_3$ is below zero, and that the true physical melting point of $(\text{H}_2\text{O})_2$ is above zero, from which it might be inferred that the melting of ice at zero is the melting of $(\text{H}_2\text{O})_3$, containing a

¹ Nature, Vol. 78, p. 382.

² Phil. Magazine, Vol. 50, pp. 460-489.

certain amount of $(\text{H}_2\text{O})_2$. When freezing is progressing the $(\text{H}_2\text{O})_2$ mixed with the $(\text{H}_2\text{O})_3$ is either very small or entirely absent but when the ice warms up to approximately zero a part of the $(\text{H}_2\text{O})_3$ is changed to $(\text{H}_2\text{O})_2$ and the mixture begins to melt.

This possible explanation lacks experimental proof and must be taken as a possible reason rather than as an actual one.

Hess¹ in experiments on ice under different observed conditions of temperature and pressure finds as Tammann previously had observed that for a given temperature a considerable movement of the plunger occurs at a pressure lower than that figured by thermodynamical calculations, a portion of the ice melting before the total calculated pressure required to melt it was applied.

The results given in this paper may be a special case of the same phenomenon for one atmosphere of pressure and zero degrees.

SUMMARY.

The results of the experiments described establish the following fact: Dry ice sticks and crystals inside a test tube which is immersed in a mixture of ice and water will melt at the same time that other ice crystals are freezing out of the mixture surrounding the tube, the crystals inside the tube and the mixture surrounding it, both being under atmospheric pressure and the temperature of the room below being 0°C . Experimental efforts to explain the results are all negative. The suggested possible explanations lack experimental proof and are possible rather than actual reasons. The facts are accordingly presented by the author with the frank admission that he has at present no satisfactory explanation to offer for them but that he deems the facts as such to be sufficiently interesting and unexpected as to warrant their publication.

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¹ Ann. Physik., Vol. 36, pp. 449-493.

SOME EMISSION THEORIES OF LIGHT.

BY RICHARD C. TOLMAN.

THE Einstein theory of relativity assumes as its second postulate, that the velocity of light is independent of the relative motion of the source of light and the observer. It has been suggested in a number of places that all the apparent paradoxes of the Einstein theory might be avoided and at the same time the principle of the relativity of motion retained, if an alternative postulate were true that the velocity of light and the velocity of the source are additive. Relativity theories based on such a postulate may well be called emission theories.

All emission theories agree in assuming that light from a moving source has a velocity equal to the vector sum of the velocity of light from a stationary source and the velocity of the source itself at the instant of emission. Thus a source in uniform motion would always remain at the center of the spherical disturbances which it emits, these disturbances moving relative to the source itself with the same velocity c as from a stationary source.¹ Emission theories differ, however, in their assumptions as to the velocity of light after its reflection from a mirror.

If an emission theory is accepted, it would seem most natural to assume that the excited portion of a reflecting mirror acts as a new source of light and that reflected light has the same velocity c with respect to the mirror as has original light with respect to its source. The possibility of such an assumption has already been suggested by the writer² and apparently disproved by an experiment on the velocity of light from the approaching and receding limbs of the sun. In the present article additional evidence disproving the possibility of the assumption will be presented.

According to an emission theory suggested by Stewart³ light reflected from a mirror acquires a component of velocity equal to the velocity of the mirror image of the original source. Evidence disproving the possibility of such a principle will also be presented in this article.

¹ Optical theories in which the velocity of light is assumed to change during the path are not considered in this article. It might be very difficult to test theories in which the velocity of light is assumed to change on passing through narrow slits or near large masses in motion, or to suffer permanent change in velocity on passing through a lense.

² Tolman, *PHYS. REV.*, 31, 26 (1910).

³ Stewart, *PHYS. REV.*, 32, 418 (1911).

A very complete emission theory of electromagnetism has been presented by Ritz.¹ According to this theory light retains throughout its whole path the component of velocity which it obtained from its original moving source, and after reflection light spreads out in spherical form around a center which moves with the same velocity as the original source. In this article an experiment will be suggested whose performance would permit a decision between the Ritz and Einstein theories of relativity.

THE FIRST EMISSION THEORY.

According to the first of the above emission theories, if a source of light is approaching an observer with the velocity v , the emitted light would have the velocity $c + v$ and after reflection from a *stationary* mirror would have the velocity c . We shall now show that measurements of the Doppler effect (in canal rays) do not agree with this theory.²

Consider measurements of the Doppler effect in light from a moving source made with a concave grating arranged as shown in Fig. 1. Light from the source (canal rays) enters the slit and falls on the grating which is so mounted that its center of curvature coincides with the position of the line of the spectrum to be photographed at D . Hence the paths BD and CD traversed after reflection by the two rays of light ABD and ACD are equal, and

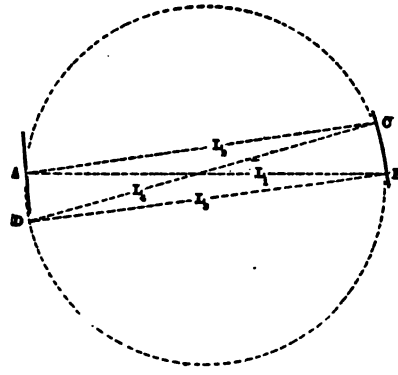


Fig. 1.

¹ Ritz, *Ann. de chim. et phys.*, 13, 145 (1908); *Arch. de Genève*, 26, 232 (1908); *Scientia*, 5 (1909). See also Gessamm. Werke. The Ritz electromagnetic theory does not seem to have received the critical attention which it deserves. It was the earliest systematic attempt to explain the Michelson Morley experiment on the basis of an emission theory and is the only emission theory which has been developed with any completeness.

² In an earlier article (*loc. cit.*), the author showed that if an emission theory of light were true, there would be no change in the wave-length of light when the source is set in motion. This undisputed conclusion led the author to believe that with a suitable arrangement of grating no Doppler effect would be detected in light from moving sources if an emission theory should be true. It has been correctly pointed out by Stewart (*loc. cit.*, p. 420), however, that the use of a grating to determine wave-lengths is based on a theory which assumes a stationary medium. Hence grating measurements of the Doppler effect do not afford a general method of testing all emission theories, but such measurements must be subjected to a more complete analysis. As shown in the sequel, however, such an analysis of existing measurements of the Doppler effect is apparently sufficient to disprove the Stewart emission theory. Such measurements are not suitable for deciding between the theories of Ritz and Einstein, however, since in general these two theories would only lead to the expectation of second order differences.

the only difference in length of path occurs, before reflection, *i. e.*, $AB = L_1 > AC = L_2$.

Consider first a *stationary source*, and let τ be the period of the source which produces a bright line at D . For the production of such a line, it is evident that light impulses coming over the two paths ABD and ACD must arrive at D in the same phase. If Δt is the time interval between the departures from the source of two light impulses which arrive simultaneously at D , the condition necessary for their arrival in phase is evidently given by the equation

$$i\tau = \Delta t = \frac{L_1 - L_2}{c}, \quad (1)$$

where i is a whole number. (Note that $L_3 = L_4$ with the apparatus as arranged.)

Consider now a source of light *approaching the slit with the velocity v* . If τ' is the period of the source which now produces a bright line at D and $\Delta t'$ the time interval between departures from the source of two light impulses which now arrive simultaneously at D we evidently have the relation

$$i\tau' = \Delta t'.$$

In order to obtain an expression for $\Delta t'$ in terms of L_1 and L_2 , we must note that the source moves toward the slit the distance $v\Delta t'$ during the interval of time between the departures of the two light impulses, and hence the difference in path which was $L_1 - L_2$ for a stationary source has now become $L_1 - L_2 + v\Delta t'$. Furthermore we must remember that according to the theory which we are investigating the light before reflection will have the velocity $c + v$,¹ and hence

$$i\tau' = \Delta t' = \frac{L_1 - L_2 + v\Delta t'}{c + v}, \quad (2)$$

$$i\tau' = \frac{L_1 - L_2}{c},$$

which by comparison with equation (1) gives us $\tau' = \tau$.

In other words if the first of the above emission theories of light is true, both before and after the source of light is set in motion, light produced by the same period of the source gives a bright line at the point D , that is, the expected Doppler effect or shifting of the lines does not occur.

In interpreting actual experimental results, it must be borne in mind that the adjustment of the grating was assumed to be such that the

¹ The slight difference in direction between the rays AB and AC and the motion of the source may be neglected.

reflected light is parallel to the axis of the grating. (Such an adjustment is automatically obtained with the Rowland form of mounting.) If the adjustment of the grating should be such that the difference in path all occurs after reflection it can easily be shown that the first theory *would* lead to a Doppler effect of the expected magnitude, and for intermediate adjustments to an effect of intermediate magnitude.

With regard to actual experimental results obtained with the reflected light parallel to the axis of the grating, the writer quotes from a letter received from Professor Stark.

Professor Stark says: "Sowohl in meinen Beobachtungen mit dem Konkav wie mit dem Plangitter (Ann. d. Phys., 28, 974, 1909) waren die gebeugten Strahlen, welche das beobachtete Spektrogram lieferten nicht parallel oder nahezu parallel der Gitteraxe. Doch hat Paschen (Ann. d. Phys., 23, 247, 1907), soviel ich sehen kann, den Doppler Effekt bei Kanalstrahlen in der Nähe der Axe (Normalen) eines Konkavgitters beobachtet; er hat dabei mit Hilfe eines Objektivs paralleles Licht auf das Gitter fallen lassen. Ein Unterschied zwischen Paschens und meinen Resultaten über den Doppler-Effekt bei Kanalstrahlen hat sich indes nicht ergeben. Die zwei Methoden (einfallendes Licht parallel der Gitteraxe, gebeugtes Licht parallel dieser) liefern also bei gleicher dispersion übereinstimmende Doppler-Effekt-Spektrogramme."

We thus see that the first of the above emission theories does not seem to accord with experimental facts.

THE STEWART THEORY.

By considering the same measurements of Doppler effect just described, it can also be shown that the Stewart theory does not agree with experimental facts.

Suppose a concave grating, Fig. 2, arranged as before with the center of curvature coinciding with the position of the line of the spectrum to be photographed at D .

Consider first a stationary source and let τ be the period of the source which produces a bright line at D . If Δt is the time interval between the departures from the source of two light impulses which after traveling over the two paths ABD and ACD arrive simultaneously at D , it is evident, as in the previous discussion that the condition necessary for their arrival in phase and hence for the production of a bright line is given by the equation

$$i\tau = \Delta t = \frac{L_1 - L_2}{c}, \quad (3)$$

where i is a whole number.

Consider now a source of light approaching the slit with the velocity v . If τ' is the period of the source which now produces a bright line at D and $\Delta t'$ the time interval between departure from the source of two light

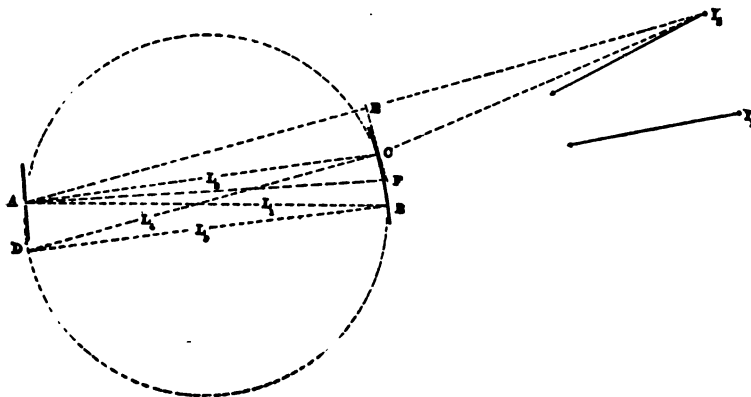


Fig. 2.

impulses which now arrive simultaneously at D , we evidently have the relation

$$i\tau' = \Delta t' = \frac{L' - L_2 + v\Delta t'}{c + v} + \frac{L_3}{c + v_3} - \frac{L_4}{c + v_4}, \quad (4)$$

where $c + v^1$ in accordance with the Stewart theory is the velocity of the light before reflection and v_3 and v_4 are the components which must be added to c to give the velocity of light along the paths $BD = L_3$ and $CD = L_4$ after its reflection.

According to the Stewart theory v_3 and v_4 will be equal to the components in the direction BD and CD of the velocities of the mirror images of the original source. An idea of the size of these components is most easily obtained graphically. Considering, for example, the point of reflection C as a portion of a plane mirror EF which is tangent to the concave mirror at C , the position of the image I_2 can be found by the usual construction, the line AI_2 connecting source and image being perpendicular to EF and the distances AE and EI_2 equal. Both the original source and the image will evidently be moving towards the point F with the same velocity v . By a similar construction, which has been omitted to avoid confusion, the image I_1 produced by reflection from B is found to be located as shown, and moves also with the velocity v in the direction of the corresponding arrow.

It can be seen from the construction that in the arrangement shown the motion of the image I_1 and the corresponding reflected ray BD are

¹ See note 1, p. 138.

more nearly parallel than the motion of I_2 and the ray CD . Hence from the principle of Stewart the component v_3 is greater than v_4 . Referring once more to equation (4), since L_3 and L_4 are equal and v_3 is greater than v_4 , we see that the negative term $L_4/(c + v_4)$ is numerically greater than $L_3/(c + v_3)$ and we may write the inequality

$$\Delta t' < \frac{L_1 - L_2 + v\Delta t'}{c + v}.$$

Neglecting second order terms this becomes

$$\Delta t' \left(1 - \frac{v}{c}\right) < \frac{L_1 - L_2}{c} - \frac{L_1 - L_2}{c} \frac{v}{c}$$

and substituting from equation (3),

$$\Delta t' \left(1 - \frac{v}{c}\right) < \Delta t \left(1 - \frac{v}{c}\right),$$

$$\Delta t' < \Delta t,$$

$$\tau' < \tau.$$

Thus on the basis of the Stewart theory, with an approaching source, a shorter period would produce a bright line at the point D than with a stationary source. In other words the actual bright lines would shift towards the red end of the spectrum when the source is set in motion towards the slit, in contradiction to the actually observed shift towards the violet end of the spectrum.

We see that experimental facts do not agree with the Stewart theory.

THE RITZ THEORY.

According to the Ritz theory of relativity, throughout its whole path, light retains the component of velocity v which it obtained from the original moving source. Thus all the phenomena of optics would occur as though light were propagated by an ether which is stationary with respect to the original source. Light coming from a terrestrial source would behave as though propagated by an ether stationary with respect to the earth and light coming from the sun would behave as though propagated by an ether stationary with respect to the sun. Now the Michelson-Morley experiment was devised for detecting the motion of the earth through the ether, and hence if this experiment should be re-performed using light from the sun instead of from a terrestrial source, a positive effect would be expected if the Ritz theory were true. On the other hand if the Einstein theory were true, no effect would be obtained, since according to this theory, all optical phenomena occur as though light were propagated by an ether stationary with respect to the observer.

To show in detail the divergence between the two theories consider the diagrammatic representation of a Michelson-Morley apparatus as shown in Fig. 3.

Light from the sun which is supposed to be moving relative to the apparatus in the direction AB with the velocity v is thrown with the help of suitable reflectors on to the half silvered mirror at A . The divided beams of light travel to the mirrors B and C and after reflection reunite at D to produce a system of interference fringes.

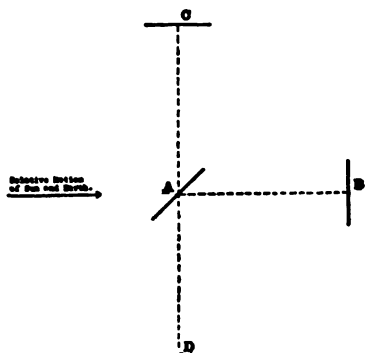


Fig. 3.

According to the Einstein theory of relativity the velocity of light is the same in all directions with respect to all observers, and hence the velocity along the paths AB and CD would be independent of the orientation of the apparatus and on the basis of this

theory no change in the position of the interference fringes would be expected on rotation of the apparatus.

According to the Ritz theory, however, the velocity of light in the directions AB and AC would be different and a change in the position of the fringes would be expected on rotating the apparatus through an angle of ninety degrees.

It is easy to see that the Ritz theory would lead us to expect $c + v$ for the velocity of light in the direction AB , $c - v$ for the velocity in the opposite direction, and $\sqrt{c^2 - v^2}$ for the velocity in either direction along AC .

Assuming for simplicity that $AB = AC = l$, we see that the time required for light to travel along the path $ABBAD$ will be longer than that along the path $ACCD$ by the amount

$$\frac{l}{c + v} + \frac{l}{c - v} - \frac{2l}{\sqrt{c^2 - v^2}},$$

which neglecting terms of higher orders reduces to lv^2/c^3 .

If the apparatus should be rotated through ninety degrees, it is evident that the longer time would now be required for the light to pass over the path $ACCD$ and we should expect a shift in the position of the fringes corresponding to the time interval

$$\frac{2lv^2}{c^3}.$$

Hence if the Ritz theory should be true, *using the sun as source of light* we should find on rotating the apparatus a shift in the fringes of the same magnitude as originally predicted for the Michelson-Morley apparatus where a terrestrial source was used. If the Einstein theory should be true, we should find no shift in the fringes using any source of light.

SUMMARY.

Experimental evidence has been considered in this article which is apparently sufficient to disprove two of the three emission theories of light which have been proposed, and an experiment has also been suggested for testing the truth of the third emission theory, that of Ritz. A definite experimental decision between the relativity theories of Ritz and Einstein is a matter of the highest importance.

The writer wishes to express his gratitude to Dr. P. Ehrenfest for valuable suggestions and criticisms, and to Professor Stark for information concerning the adjustment of his gratings in the measurement of the Stark effect in canal rays.

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THE ELECTRICAL CONDUCTIVITY OF BISMUTH-THALLIUM ALLOYS.

BY A. E. WHITFORD.

THE recent work done by Haken¹ in investigating the electrical properties of certain alloys, especially of bismuth, has stimulated interest in this line of research. This was followed by the work of Mendenhall and Lent,² who determined the magnetic susceptibility of the bismuth-tellurium and bismuth-thallium series of alloys. These investigations have confirmed in the main the idea that the maxima of the melting-point curves are evidence of chemical combination, for at these points Haken found marked discontinuities in the thermo-electric curve and the electrical conductivity curve.

The present investigation was undertaken to study further the extent to which compounds as determined from the melting-point curve influence the trend of other physical properties of alloys. The specimens were cast in glass tubing in an atmosphere of hydrogen and when finished were from one to three centimeters long and about two millimeters in diameter. The metals to be fused were placed in the bottom of a small test tube which was supported vertically in an electric furnace. Two glass tubes were inserted in the cork stopper of the test tube. Through one tube the hydrogen entered the test tube and through the other, to which was attached a three-way stopcock, the hydrogen escaped to the air. The second tube however served a double purpose. It was adjustable so that it could be raised or lowered in the test tube, and by turning the three-way stopcock, an aspirator served to draw the fused metal into the glass tube. To insure a homogeneous mixture in the preparation of the alloy, in many cases the metals were kept in a molten condition in the bottom of the test tube from fifteen to thirty minutes and the test tube was frequently shaken. Especially was great care used in this respect for those mixtures whose composition was at or near the compound Bi_3Tl_2 . When a cast was to be made the adjustable glass tube was lowered into the molten metal and the aspirator was started up very slowly. As soon as about 3 cm. of the metal were drawn up in the glass

¹ Haken, *Ber. d. Deutsch. Gesell.*, March 15, 1910; *Annalen der Physik*, 32, 291, 1910.

² *Phys. Rev.*, XXXII., No. 4, April, 1911.

tube, the stopcock was given a quarter turn, the test tube was lifted out of the furnace and the specimen thus formed was allowed to cool in the air. Inasmuch as thallium contracts somewhat on cooling, all specimens except those of pure bismuth were forced out of the glass tubes in which they were cast, by simply pushing at one end with a small tool. In the case of 100 per cent. bismuth, the glass was removed by hydrofluoric acid.

The bismuth used was Kahlbaum's chemically pure; the thallium was Schuckhardt's chemically pure. Since the composition calculated from the weights originally put into the alloys are known to check very closely with chemical analyses (Monkmeyer and Chikashige),¹ the alloys were not analyzed after the measurement of their conductivities.

The resistance of each specimen was determined by a Leeds and Northrup potentiometer. The specimen rested on two brass knife edges which were so arranged on a block that the distance between them could be varied. Uniform pressure of the specimens on the knife edges was secured by suspending the same weight from the specimen for each determination. The resistance measured varied from 0.01 ohms to 0.001 ohms and could be determined accurately to the fifth decimal place.

The accompanying table of results shows the electrical conductivity of each specimen and its method of preparation. Figure 1 shows these results graphically. There are added to the figure the melting-point curve from Monkmeyer and Chikashige,¹ and the susceptibility curve from Mendenhall and Lent.²

No satisfactory explanation was found for the cause of the variations in the value of the conductivity for a given percentage of mixture. This error seems to be greatest for Bi_3Tl_3 and probably is here partially explained by the fact that these points are on the steep maximum of the curve and the assumed composition may have been slightly in error. The results given are all derived from specimens which were chilled in the air immediately after casting. Some of these were afterwards annealed, but the results did not justify any conclusion as to the cause of the errors in the conductivity. It may be noted however that these variations are no greater than those in Haken's³ determination of the conductivity of the bismuth-tellurium alloys.

As was to be expected, at 63 per cent. Bi corresponding to Bi_3Tl_3 , the electrical conductivity curve shows a discontinuity, but the curve does

¹ *Zeitschrift f. anorg. Chem.*, 46, 415, 1905, and 51, 328, 1906.

² *Loc. cit.*

³ *Annalen der Physik*, 32, 291, 1910.

TABLE.
Bismuth-Thallium Alloys.

Specimen Number.	Per Cent. Bi Weight.	Method of Preparation.	El. Cond. $\times 10^7$.
7	00.00	Cut directly from stick	529.6
18	00.00	Cut directly from stick	539.5
24	10.00	From part of No. 18 by adding Bi.	259.0
25	10.00	From remainder of No. 18 by adding Bi.	259.3
48	10.90	From No. 17 by adding Tl.	229.8
51	10.90	No. 48 remelted	228.5
10	20.00	From No. 7 by adding Bi.	159.3
11	20.00	No. 10 remelted	154.3
26	20.00	From No. 25 by adding Bi.	149.8
27	20.00	No. 26 remelted	152.5
17	30.00	From No. 14 by adding Tl.	104.6
28	30.00	From No. 27 by adding Tl.	104.5
16	40.00	Mixing No. 28 and No. 14.	101.8
29	40.00	From No. 28 by adding Bi.	100.1
9	50.00	From No. 7 by adding Bi.	102.1
12	50.00	From No. 9 and a fresh mixture.	110.2
30	50.00	From No. 29 by adding Bi.	107.5
33	50.00	From No. 16 by adding Bi.	107.0
14	60.00	From No. 11 by adding Bi.	128.7
31	60.00	From No. 30 by adding Bi.	128.2
54	61.50	Melting together	134.5
19	62.50	Melting together	139.8
22	62.50	Melting together	142.7
32	63.00	From No. 31 by adding Bi.	147.3
35	63.00	Melting together	152.5
44	63.00	Melting together	160.2
46	63.00	Melting together	152.3
47	63.00	Melting together	154.5
50	64.95	Mixing No. 44 and No. 49.	159.5
49	66.00	From No. 17 by adding Bi.	157.9
34	70.00	From No. 16 by adding Bi.	150.5
38	70.00	From No. 37 by adding Bi.	156.6
41	70.00	Melting together	151.5
8	80.00	Melting together	114.3
39	80.00	From No. 38 by adding Bi.	108.1
42	80.00	From No. 41 by adding Bi.	112.2
4	90.00	Melting together	99.04
5	90.00	Melting together	93.45
40	90.00	From No. 38 by adding Bi.	93.60
43	90.00	From No. 42 by adding Bi.	96.7
1	100.00	Cut directly from stick	85.2
2	100.00	Cut directly from stick	83.6
3	100.00	Cut directly from stick	83.35

not form an ordinary cusp at this point, since its slope is steep only on one side of the point. At the other bismuth-thallium compound (10.9 per cent. Bi) no discontinuity was detected. The characteristics of this

curve agree well with those of the magnetic susceptibility curve of the bismuth-thallium series, where a well-formed cusp appears at Bi_6Tl_4 , but no discontinuity is evident at the other compound point (Fig. 1).

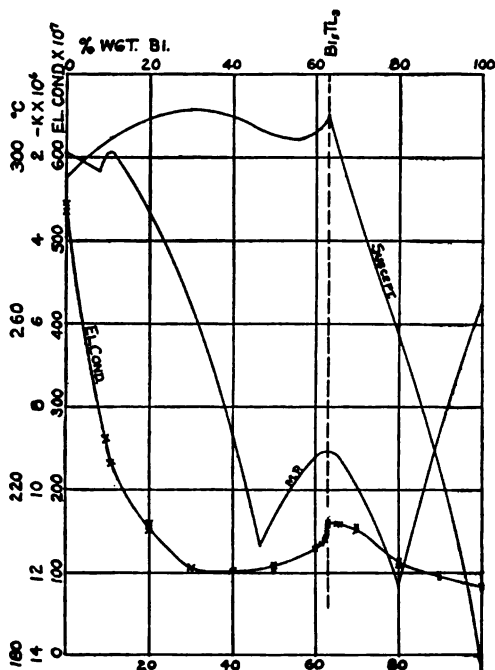


Fig. 1.

The compound containing 10.9 per cent. Bi, while it is of such a nature as to affect the melting-point curve, does not affect the conductivity or susceptibility curves.

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NOTE ON ABSOLUTE FORMULÆ FOR THE MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

By G. R. OLSHAUSEN.

1. In a previous paper¹ it was shown that in the formula for the mutual inductance of two coaxial solenoids

$$(1) \quad M = 4\pi n n' [I_1 - I_2 - I_3 + I_4]$$

the terms in the brackets can be put into the form

$$(2) \quad I = \frac{(m\beta)^{\frac{1}{2}}}{2} \left[\eta_1 p w + \left\{ \frac{g_3}{6} - p^2 w + \left(\frac{p'w}{2} \right)^2 \frac{1}{pw - e_3} \right\} \omega_1 + \frac{p'w}{2} \left\{ \eta_1 w_2 i - \omega_1 \frac{\sigma_1'}{\sigma_1} (w_2 i) - \frac{1}{2} \pi i \right\} \right].$$

In this expression w_2 was determined by the relations

$$w = \omega_2 + w_2 i, \quad 0 < w_2 < \frac{\omega_2}{i},$$

w satisfying the known values of pw and $p'w$ when the latter is taken with its negative sign.

In the paper cited the expression (2) had not been put into a convenient form for computation, but had only been used to derive Cohen's formula.

It is the object of this paper to adapt the expression (2) to computation and to give some additional formulæ for calculating the invariant g_3 .

2. If

$$w = \omega_1 + w_1 i, \quad 0 < w_1 < \frac{\omega_1}{i}$$

is the value of w which satisfies the values of pw and $p'w$ when $p'w$ is taken with its positive sign, then, since pw is an even elliptic function of the second order having the periods $2\omega_1$ and $2\omega_2$, it is easy to see that

$$(3) \quad iw_2 = \omega_2 - iw_1.$$

By means of this relation and the expressions

¹ G. R. Olshausen, *Phys. Rev.*, p. 617, Vol. 31, Dec., 1910.

$$(4) \quad \begin{aligned} \frac{\sigma_1'}{\sigma_1}(u - \omega_1) &= \frac{\sigma'}{\sigma}(u) - \eta_1, \\ \frac{\sigma'}{\sigma}(u + \omega_2) &= \frac{\sigma_2'}{\sigma_2}(u) + \eta_2 \end{aligned}$$

the last term of (2) can be expressed in terms of w_1 .

If in the first and second equations of (4) we let

$$u = \omega_2 - w_1 i$$

and

$$u = -w_1 i$$

respectively and then combine, we obtain

$$(5) \quad \frac{\sigma_1'}{\sigma_1}(w_1 i) = \frac{\sigma_1'}{\sigma_1}(\omega_2 - w_1 i) = \eta_2 - \frac{\sigma_2'}{\sigma_2}(w_1 i).$$

Substituting (3) and (5) in the last term of (2) and noting that

$$\eta_1 \omega_2 - \omega_1 \eta_2 = \frac{1}{2} \pi i,$$

we get

$$(6) \quad I = \frac{(m\beta)^{\frac{1}{2}}}{2} \left[\eta_1 p w + \left\{ \frac{g_2}{6} - p^2 w + \left(\frac{p' w}{2} \right)^2 \frac{1}{p w - e_2} \right\} \omega_1 - \frac{p' w}{2} \left\{ \eta_1 w_1 i - \omega_1 \frac{\sigma_2'}{\sigma_2}(w_1 i) \right\} \right],$$

where $p'w/2$ is to be taken with its negative sign.

By employing the relations existing between the σ -functions and the ϑ -functions we have, when

$$e_2 < 0,$$

$$(7) \quad \begin{aligned} \eta_1 w_1 i - \omega_1 \frac{\sigma_2'}{\sigma_2}(w_1 i) \\ = i\pi \left\{ \frac{h(z^2 - z^{-2}) + 2h^4(z^4 - z^{-4}) + 3h^9(z^6 - z^{-6}) + \dots}{1 + h(z^2 + z^{-2}) + h^4(z^4 + z^{-4}) + h^9(z^6 + z^{-6}) + \dots} \right\} \end{aligned}$$

and when

$$e_2 > 0,$$

$$(8) \quad \begin{aligned} \eta_1 w_1 i - \omega_1 \frac{\sigma_2'}{\sigma_2}(w_1 i) \\ = i\pi \left\{ v - \frac{i\omega_1}{\omega_2} \cdot \frac{2h_1 \sin 2\pi v + 4h_1^4 \sin 4\pi v + 6h_1^9 \sin 6\pi v + \dots}{1 + 2h_1 \cos 2\pi v + 2h_1^4 \cos 4\pi v + 2h_1^9 \cos 6\pi v + \dots} \right\}. \end{aligned}$$

The absolute value of the expressions (7) and (8) is smaller than that of the corresponding expressions (52) and (59) of the former paper,¹ the symbols having the same significance.

3. By a proper choice of the parameter m it will always be possible to calculate accurately the value of $g_2/6$ from the expression

¹ Loc. cit.

$$(9) \quad \frac{g_2}{6} = \frac{2}{9m^2} \left[\left(\frac{a_1^2 + a^2 + c^2}{2a_1a} \right)^2 + 3 \right].$$

This quantity may, however, also be computed by means of the ϑ -functions as follows. We have, when

$$e_2 < 0,$$

$$(10) \quad \begin{aligned} \frac{g_2}{6} &= \frac{1}{9} \left(\frac{\pi}{2\omega_1} \right)^4 [\vartheta_2^3(0) + \vartheta_3^3(0) + \vartheta_0^3(0)] \\ &= \frac{1}{9} \cdot \frac{1}{4^4} \left[\frac{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2}}{1 + h^4 + 2h^{16} + \dots} \right]^8 [\vartheta_2^3(0) + \vartheta_3^3(0) + \vartheta_0^3(0)], \end{aligned}$$

where

$$\vartheta_2(0) = 2h^{\frac{1}{2}}(1 + h^2 + h^6 + h^{12} + \dots),$$

$$\vartheta_3(0) = 1 + 2h + 2h^4 + 2h^9 + \dots,$$

$$\vartheta_0(0) = 1 - 2h + 2h^4 - 2h^9 + \dots.$$

On substituting these expressions in (10) and expanding, we obtain

$$\begin{aligned} \frac{g_2}{6} &= \frac{1}{2304} \left[\frac{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2}}{1 + 2h^4 + 2h^{16} + \dots} \right]^8 \\ &\quad \times [2 + 480h^2 + 4320h^4 + 13440h^6 + 35040h^8 \dots] \end{aligned}$$

or

$$(11) \quad \frac{g_2}{6} = \frac{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_2 - e_3})^8}{1152} \times (1 + 240h^2 + 2144h^4 + 2880h^6 - 16896h^8 \dots).$$

When

$$e_2 > 0,$$

$$(12) \quad \begin{aligned} \frac{g_2}{6} &= \frac{1}{9} \left(\frac{\pi i}{2\omega_1} \right)^4 [\vartheta_2^3(0 | \tau_1) + \vartheta_3^3(0 | \tau_1) + \vartheta_0^3(0 | \tau_1)] \\ &= \frac{1}{9} \cdot \frac{1}{4^4} \left[\frac{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_2 - e_3}}{1 + 2h_1^4 + 2h_1^{16} \dots} \right]^8 [\vartheta_2^3(0 | \tau_1) + \vartheta_3^3(0 | \tau_1) + \vartheta_0^3(0 | \tau_1)], \end{aligned}$$

where

$$\vartheta_2(0 | \tau_1) = 2h_1^{\frac{1}{2}}(1 + h_1^2 + h_1^6 + h_1^{12} + \dots),$$

$$\vartheta_3(0 | \tau_1) = 1 + 2h_1 + 2h_1^4 + 2h_1^9 + \dots,$$

$$\vartheta_0(0 | \tau_1) = 1 - 2h_1 + 2h_1^4 - 2h_1^9 + \dots.$$

On substituting these values and expanding, we obtain in the same manner as before

$$(13) \quad \frac{g_2}{6} = \frac{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_2 - e_3})^8}{1152} \times (1 + 240h_1^2 + 2144h_1^4 + 2880h_1^6 - 16896h_1^8 \dots).$$

It is to be noted that when $e_2 \leq 0$, $h \leq e^{-\pi} < 1/23$ and when $e_2 \geq 0$, $h_1 \leq e^{-\pi} < 1/23$, so that the terms $16896h^2$ and $16896h_1^2$, are then both less than 2×10^{-7} .

4. Nagaoka¹ has also employed the ϑ -functions to determine $g_2/6$, but his formulæ do not converge very rapidly when $\sqrt{k'}$ is small. This occurs when in the expression

$$k'^2 = \frac{(a_1 - a)^2 + c^2}{(a_1 + a)^2 + c^2}$$

the numerator is small compared to $(a_1 + a)^2$, for example, when two nearly concentric coils of the same length have nearly the same radius.

For the case computed in the paper referred to above, where

$$d = 0, 2l = 200, 2h_1 = 20, a = 10, a_1 = 15,$$

the results of the computation are given below.

In this computation $g_2/6$ was calculated by equation (9), the parameter m having been so chosen as to make $e_1 - e_3 = 1$.

The quantities

$$l = \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} \text{ and } l' = - \frac{1 - \sqrt{\frac{e_1 - s}{k'}}}{1 + \sqrt{\frac{e_1 - s}{k'}}}$$

were computed by means of the relation

$$\frac{1 - \cos x}{1 + \cos x} = \tan^2 \frac{x}{2}$$

by letting $\cos x$ equal $\sqrt{k'}$ and $\sqrt{\frac{e_1 - s}{k'}}$ respectively.

If equation (11) is used to compute $g_2/6$, we find

$$\frac{g_2}{6} = 0.21223833 \text{ for } c = 110,$$

$$\frac{g_2}{6} = 0.20799141 \text{ for } c = 90$$

and

$$M = 4\pi nn'6213.501$$

instead of

$$M = 4\pi nn'6213.428$$

as given below.

The values of $g_2/6$ found by using (9) are the better ones, however, and were checked by means of Crelle's multiplication tables.

On recalculating M by (21) and (61)² with greater precision than before,

¹ H. Nagaoka, Jour. Coll. Sci., Tokyo, 27, art. 6, 1909.

² G. R. Olshausen, l. c.

we find

$$M = 4\pi n n' 6213.482$$

instead of

$$M = 4\pi n n' 6213.77.$$

	$c = 110.$	$c = 90.$
$\log k'^2 = \log \frac{(a_1 - a)^2 + c^2}{(a_1 + a)^2 + c^2}$	9.9790239	9.9690580
$\log \sqrt{k'} = \log \cos x$	9.9947560	9.9922645
$\log l = \log \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} = \log \tan^2 \frac{x}{2}$	7.7808467	7.9496635
$\log h = \log \left(\frac{l}{2} + \dots \right)$	7.4798167	7.6486335
$\log \omega_1 = \log \frac{2\pi}{(1 + \sqrt{k'})^2} (1 + \dots)^2$	0.2013480	0.2038209
$\log \eta_1 = \log \frac{\pi^2}{12\omega_1} \frac{1 - 27h^2 \dots}{1 - 3h^2 \dots}$	9.7136755	9.7110910
$\log pw = \log \frac{1}{3} \frac{2(a_1^2 + a^2) - c^2}{(a_1 + a)^2 + c^2}$	9 _n .4770264	9 _n .4542696
$\log \sqrt{e_1 - s} = \log \sqrt{\frac{c^2}{(a_1 + a)^2 + c^2}}$	9.9890638	9.9838598
$\log \sqrt{\frac{e_1 - s}{k'}} = \log \cos x'$	9.9943078	9.9915953
$\log l' = \log \tan^2 \frac{x'}{2}$	7 _n .8164594	7 _n .9856954
$-l'$	1.0854573	1.0865056
$\log hz^2 = \log (-\sqrt{l'}^2 - 1 - l')$	9.8217095	9.8206346
$\log \left\{ \frac{h(z^2 - z'^2) + \dots}{1 + h(z^2 + z'^2) + \dots} \right\}$	9.6007265	9.6000650
$\log \frac{p'w}{2i} = \log \frac{(a_1^2 + a^2)c}{(a_1 + a)^2 + c^2}$	7 _n .9813160	8 _n .1400043
$\log (pw - e_3) = \log \frac{(a_1 + a)^2 + c^2}{(a_1 + a)^2 + c^2}$	8.6912222	8.8551146
$\log \frac{p'w}{2} \left\{ \eta_1 w_1 i - \omega_1 \frac{\sigma_1'}{\sigma_2} (w_1 i) \right\}$	8.0791924	8.2372192
$\frac{g_2}{6} = \frac{2}{9} \left[\left(\frac{a_1^2 + a^2 + c^2}{(a_1 + a)^2 + c^2} \right)^2 + 3 \left(\frac{2a_1 a}{(a_1 + a)^2 + c^2} \right)^2 \right]$	0.21223822	0.20799139
$\left(\frac{p'w}{2} \right)^2 \frac{1}{pw - e_3}$	0.08996069	0.08101011
$\lambda = \omega_1 \left\{ \frac{g_2}{6} - p^2 w + \left(\frac{p'w}{2} \right)^2 \frac{1}{pw - e_3} \right\}$	-0.00186814	-0.00266012
$\mu = -\frac{p'w}{2} \left\{ \eta_1 w_1 i - \omega_1 \frac{\sigma_1'}{\sigma_2} (w_1 i) \right\}$	0.19142930	0.19877689
$\nu = pw \cdot \eta_1$	-0.01200031	-0.01726709
$\lambda + \mu + \nu$	-0.15513219	-0.14633915
$\log (\lambda + \mu + \nu)$	0.02429681	0.03517065
$\log (m\beta) = \log [(a_1 + a)^2 + c^2]$	8.3855510	8.5461804
$\log 2I = \log (m\beta) (\lambda + \mu + \nu)$	6.1569867	5.9111482
$2I$	4.5425377	4.4573286
	34876.887	28663.459

$$2(I_1 - I_2) = 2(I_4 - I_5) = 34876.887 - 28663.459 = 6213.428,$$

$$M = 4\pi n n' 6213.428.$$

THE PHYSICAL REVIEW.

ON THE THEORY OF RELATIVITY: ANALYSIS OF THE POSTULATES.

BY R. D. CARMICHAEL.

INTRODUCTION.

THIS analysis of the postulates¹ of relativity was undertaken in order to ascertain on just which of the postulates certain fundamental conclusions of the theory depend. A moment's reflection will convince one of the importance of such an analysis. Some of the conclusions of relativity have been attacked by those who admit just the parts of the postulates from which the conclusions objected to can be derived by purely logical processes. In this paper I have sought to establish some of the most fundamental and most readily accessible conclusions of the theory on the smallest possible foundation from the postulates. This plan of treatment, instead of giving rise to more complicated arguments than those hitherto usually employed, has had the opposite effect of leading to increased simplicity both in the notions which enter and in the arguments by which proofs are reached.

When the work was taken in hand it soon became evident that there was something to be done both on the postulates themselves and on the very first theorems which are to be deduced from them, as the reader will see by reference to the treatment below. It thus appears that some of the most striking conclusions of the theory depend on only a part of the postulates. To bring this fact prominently into view in one's mind is to put the whole subject in a clearer light where one may see better the interactions of its parts and its general relations to the whole body of scientific and philosophical knowledge.

¹ In the theory of relativity the word "postulate" has been used in the sense in which one is accustomed to employ the term "law of nature."

A certain method¹ has been consistently employed throughout the paper to indicate the postulates on which each theorem depends. Each postulate is designated by a letter. At the end of a theorem and enclosed in parentheses are references (by means of these letters) to the postulates on which the theorem as demonstrated depends. Thus theorem I. depends on postulates M and R' .

In carrying out the initial purpose of the paper an important part of the general fundamentals of the theory of relativity come in for a fresh development along lines more or less new. It was observed that the addition to this essential matter of a relatively small amount of material would make the paper as a whole serve as an elementary introduction to the entire theory; and consequently such matter as was necessary to this end has been incorporated. I felt the more disposed to do this in view of the fact that even this additional material is presented in a somewhat novel manner. It is believed that the paper in its present form may be read profitably by one who is making his first acquaintance with the theory and that it will afford an easier introduction than any which has yet been offered.

In every body of doctrine which consists of a finite number of postulates and their logical consequences there are necessarily certain theorems which have the following fundamental relation to the whole body of doctrine: By means of one of these theorems and all the postulates but one that remaining postulate may be demonstrated. That is, one may *assume* such a theorem in place of one of the postulates and then demonstrate that postulate. When the postulate has thus been proved it may be used in argument as well as the theorem itself; hence it is clear that all the consequences which were obtained from the first set of postulates may now be deduced again, though perhaps in a somewhat different manner. That is, if we consider the whole body of doctrine, composed of postulates and theorems, this totality is the same in the two cases. Two sets of postulates which thus give rise to the same body of doctrine (consisting of postulates and theorems together) are said to be logically equivalent.

The problem of the logical equivalents of a given set of postulates is readily seen to be an important one. Not all sets of postulates logically equivalent to a given one are equally interesting. In fact, some sets may be cumbersome in form and unsatisfactory from an æsthetic point of view. In the present analysis of the postulates of relativity attention has been given to determining some of their important logical equivalents—especial attention being given to those postulates which may replace the so-called

¹ This method has been employed by Veblen and Young in their *Projective Geometry*, 1910.

second postulate of relativity (our postulate *R*). A remark in this direction has already been made by Tolman.¹ An indication of the results of this character obtained in the present paper is given below in the description of the contents of part II. The principal value of such a matter, from the point of view of physical science, consists in the fact that it affords alternative methods for the experimental proof or disproof of the theory of relativity and that it emphasizes in an effective way the essential difficulties and limitations of such experimental verification in general.

It is the writer's purpose to treat further in a future paper the matter of the logical equivalents of the postulates. In this forthcoming work it is the intention to introduce the general laws of conservation of energy, mass, momentum and electricity, to deduce certain joint consequences of these laws and the principle of relativity, and to determine which of the theorems so obtained may replace the relativity postulates (or one of them) without destroying the equivalence of the resulting totality of doctrine.

Part I. of the present paper is devoted to a general statement and preliminary analysis of the postulates of relativity. In § 1 I give the fundamental homogeneity postulates of space and time which underlie all physical theory. In § 2 the first characteristic postulate of relativity is given, while § 3 contains a statement of the second postulate. It is shown that a part of this postulate (in the form in which the postulate is usually stated) is a consequence of the other part and of the first postulate, together with the fundamental homogeneity postulates.

Writers on relativity have usually stated only these two postulates. But as a matter of fact every one has made further assumptions; in some cases it appears to have been done unconsciously. To the present writer it seems desirable that these assumptions should be brought into the light as postulates. Accordingly, in § 4, I give those additional postulates which I shall use. One familiar with the theory will see that these assumptions are different from those usually employed (without explicit statement as postulates), as by Einstein² for instance. The choice has been made in the interest of simplicity in postulates and in proofs. The writer believes that this innovation in the statement of the postulates is important. It leads to new and simpler proofs than those ordinarily employed; and this, it is hoped, will in large measure remove the feeling of vagueness which many persons experience in approaching the theory of relativity for the first time.

¹ PHYSICAL REVIEW, 31 (1910): 26-40.

² Jahrbuch der Radioaktivität, 4 (1907): 411-462. See the assumptions stated in a footnote on p. 420.

Further remarks (of a general nature) on the postulates are added in §§ 5 and 6.

In part II. I give a discussion of the relative measurements of space and time in two systems of reference which move with respect to each other and obtain Einstein's formulæ of transformation. In §§ 7 and 8 it is shown that the most remarkable part of the conclusions of relativity concerning the time and space units is due to a part of the second postulate along with the other postulates; compare theorems III. and IV. In § 9 I treat the question of simultaneity of events happening at different places. In §§ 10 and 11 I obtain Einstein's formulæ of transformation from one system of reference to another and also the addition theorem of velocities. These results are applied in § 12 to the problem of finding logical equivalents for the postulate *R*.

I. THE POSTULATES OF RELATIVITY.

§ 1. *Postulates of Homogeneity*.—There are two fundamental postulates concerning the nature of space and time which underlie all physical theory. They assert in part that every point of space is like every other point and that every instant of time is like every other instant. For our present purpose these postulates can best be given the following more exact and complete statement.

POSTULATE H_1 . *Space is homogeneous and three-dimensional.*

POSTULATE H_2 . *Time is homogeneous and one-dimensional.*

One important meaning of these postulates, mathematically, is that the transformations of the space and time coördinates are to be linear.

All our theorems will depend directly or indirectly on these two postulates, those concerning space depending on H_1 and those concerning time depending on H_2 . Moreover, it is certain that no one will be disposed seriously to call these postulates in question. On account of these facts we shall consider it unnecessary to give any explicit reference to these postulates as part of the basis on which any particular theorem depends, it being understood once for all that they underlie all our work.

§ 2. *The First Characteristic Postulate*.—Those who postulate the existence of an ether as a means of explaining the facts about light, electricity and magnetism have usually been in general agreement¹ as to the conclusion that this ether is stationary. Experimental facts, which have to be accounted for, cannot be explained satisfactorily on the hypothesis of a mobile ether. The theory of a stationary ether leads naturally to the conclusion that it would be possible for an observer to detect and measure

¹ See, however, a presentation of the opposing view by H. A. Wilson, *Phil. Mag.*, 19 (1910): 309-817.

his absolute motion with respect to the ether. In this way it was predicted that the time which would be required for a beam of light to pass a given distance and return would be different in the two cases when the path of light was parallel to the direction of motion and when it was perpendicular to the direction of motion.

But a classical experiment of Michelson and Morley,¹ in which the ray-path was wholly in air, put this prediction to a crucial test; and not the slightest difference of time was found in the passage of light along the two paths. The extreme precision of their methods leaves no doubt as to the accuracy of the results.

In a similar manner the theory of a stationary ether gave rise to the prediction that a charged condenser suspended by a wire would exhibit a torsional effect due to the earth's motion. This prediction was tested in the crucial experiment of Trouton and Noble² with the result of showing that no such torsional effect is present.

These results are in perfect agreement with the hypothesis that it is impossible to detect absolute translatory motion through space; and as a matter of fact they have been generalized into this hypothesis. A sharp formulation of this conclusion constitutes the first characteristic postulate of relativity.

Before stating the postulate, however, it will be necessary to introduce a definition. In order to be able to deal with such quantities as are involved in the measurement of motion, time, velocity, etc., it is necessary to have some system of reference with respect to which measurements can be made. Let us consider any set of things consisting of objects and any kind of physical quantities³ whatever each of which is at rest with reference to each of the others. Let us suppose that among these objects are clocks, to be used for measuring time, and rods or rules, to be used for measuring length. Such a set of objects and quantities, at rest relatively to each other, together with their units for measuring time and length, we shall call a system of reference.⁴ Throughout the paper we shall denote such a system by S . In case we have to deal at once with two or more systems of reference we shall denote them by $S_1, S_2, \dots$. Furthermore, it will be assumed that the units of any two systems S_1 and S_2 are such that the same numerical result will be obtained in measuring with the units of S_1 a quantity L_1 and with the units of S_2 a quantity L_2 when the relation of L_1 to S_1 is precisely the same as that of L_2 to S_2 .

¹ Am. Journ. Science (3), 34 (1887): 333-345.

² Phil. Trans. Roy. Soc. (A), 202 (1904): 165.

³ As, for instance, charges, magnets, light-sources, telescopes, etc.

⁴ If any number of these objects or quantities are absent we shall sometimes refer to what remains as a system of reference. Thus the system might consist of a single light-source alone.

With this definition before us we are now able to state the first characteristic postulate of relativity:

POSTULATE *M*. *The unaccelerated motion of a system of reference S cannot be detected by observations made on S alone, the units of measurement being those belonging to S.*

The postulate, as stated, is a direct generalization from experiment. None of the actually existing experimental evidence is opposed to it. The conviction that future evidence will continue to corroborate it is so strong that objection has seldom or never been offered to this postulate by either the friends or the foes of relativity. No means at present known will enable the observer to detect absolute motion or motion through any sort of medium which may be assumed to pervade space. Furthermore, in every case where the heretofore accepted theory has predicted the possibility of detecting such motion and where sufficiently exact observations have been made, it has turned out that no such motion was detected. Moreover, one at least¹ of these contradictions of theory has been outstanding for a period of twenty-five years and no satisfactory explanation has been offered unless one is willing to accept the law stated in postulate *M* above. It would appear, therefore, that in the present state of knowledge, the experimental evidence for the postulate should be considered of strong character.

One additional remark should be made here. The direct experimental evidence which led to the formulation of postulate *M* was undertaken on account of predictions made on the basis of a theory of the ether as the vehicle of light and electricity. But the result which has been obtained is of a purely experimental character and does not in any way depend on a theory of the ether. In other words, the law stated in postulate *M* is in no way dependent either on the existence or the non-existence of the ether. It is important to keep this in mind on account of the confusion of thought which has arisen in some quarters as to the relation between the theory of relativity and the theory of the ether. The postulate is simply a generalization of experimental fact; and, unless an experiment can be devised to show that this generalization is not legitimate, it is natural and in accordance with the usual procedure in physics to accept it as a "law of nature." Theory, then, must be made to agree with it and not it with theory.

§ 3. *The Second Characteristic Postulate.*—The so-called second postulate of relativity, in the form in which it has frequently been stated,² involves two entirely distinct parts. To the present writer it appears

¹ Reference here is to the Michelson and Morley experiment.

² See the postulate *R* below and the remarks which lead up to it.

that no inconsiderable part of the difficulty which has been felt concerning this second postulate has been due to a failure to perceive the interdependence of these two parts and of postulate *M* above. Precisely that part of the second postulate to which most objection has been raised is a logical consequence of *M* and of the other part, the part last mentioned being a statement of a law which for a long time has been accepted by physicists. Consequently, we shall state separately the two parts of the second postulate and bring out with care the interdependence of these and postulate *M* above.

The part which we shall give first states a principle which has long been familiar in the theory of light, namely, that the velocity of light is unaffected by the velocity of the source. In other words the velocity with which light passes an observer is not increased if the light-source moves toward the observer or decreased if it recedes from him. Stated in exact language this postulate is as follows:

POSTULATE *R'*. *The velocity of light in free space, measured on an unaccelerated system of reference S by means of units belonging to S, is independent of the unaccelerated velocity of the source of light.*

The law stated in this postulate is a conclusion which follows readily from the usual undulatory theory of light and will therefore be accepted by any one who holds to that theory. But it should be emphasized that *R'* does not depend for its truth on any theory of light. It is a matter for direct experimental verification or disproof, and this should be made in such way as to be independent, as far as possible, of all general theories of light, at least insofar as they are not supported by *direct* experimental evidence. So far as the writer is aware, there is no experimental evidence which is undoubtedly opposed to postulate *M*, while on the other hand there is direct experimental evidence which by some is believed to be definitely in its favor. Tolman,¹ in particular, has considered this matter in relation to the Doppler effect and to the velocity of light from the two limbs of the sun; and has concluded that experiment bears out the postulate. Stewart,² on the other hand, has examined the same experiments and has found an explanation for them in Thomson's electromagnetic emission theory of light. According to Stewart these experiments are in agreement with our postulate *M* but are opposed to our postulate *R'*. All other attempted proof or disproof of the postulate appears to be in the same state; it is capable of two interpretations which are directly opposed to each other with respect to their conclusions as to the validity of *R'*. Thus at present there is no undoubted experimental

¹ PHYSICAL REVIEW, 31 (1910): 26-40.

² PHYSICAL REVIEW, 32 (1911): 418-428.

evidence for or against postulate R' . If the assumption is to be proved at all, either new experiments must be devised or it must be proved by indirect means by showing that it is a consequence of experiment and accepted laws.

Now any one who accepts postulates M and R' will perforce accept also all the logical consequences which necessarily flow from them. Of these logical consequences we shall now prove one which is of great importance in the theory of relativity:

THEOREM I. *The velocity of light in free space, measured on an unaccelerated system of reference S by means of units belonging to S , is independent of the direction of motion of S (MR').¹*

Since by R' the velocity of light is independent of that of the light-source we may suppose that the light-source belongs to the system S . Now let the velocity of light, as it is emitted from this source in various directions, be observed and tabulated. On account of the homogeneity of space mere direction through space will have no effect on these observed velocities; and therefore if they differ at all, the difference will be due to the velocity of S . Now if there were a difference due to the direction of motion of S this difference would put in evidence the motion of S . But by M it is impossible to detect such motion in this way. Hence the observed velocity must be the same in all directions. In other words, it is independent of the direction of motion of S ; and thus the theorem is proved.

It is clear, however, that we cannot take the next step and prove that this observed velocity of light is independent of the absolute value of the velocity of S , as Tolman² appears to conclude. To see this clearly, let us suppose that the absolute value of the velocity of S does affect the observed velocity of light. On account of R' it will have the same effect on the observed velocity of light whatever may be the unaccelerated motion of the light-source. Hence, from all possible observations, the experimenter will have only a single datum from which to determine the effect of one phenomenon on another; namely, a datum in which the two phenomena are connected in a certain definite way. It is obvious then that he cannot determine the effect of one of the phenomena on the other; for he can never observe the one without the other being present also and the connection which exists between them is always the same however he may vary his experiment. And if the observer cannot determine an existing effect it is clear that he cannot prove the absence of any effect whatever.

¹ Letters attached to a theorem in this way indicate those of the postulates (exclusive of H_1 and H_2) on which the theorem depends. See Introduction and §1.

² PHYSICAL REVIEW, 31 (1910), p. 27.

But, *although the absence of this effect cannot be proved, it is probably impossible to conceive any satisfactory way in which it could be present.* Physical intuition is emphatic (and it may be that this is what Tolman intended to say in the passage cited) in asserting that if the direction of the velocity of S has no effect on the observed velocity of light then the absolute value of the velocity of S has no effect on such observed velocity. But this does not constitute a proof. There is in this, however, nothing to invalidate the naturalness of the *assumption* of such independence of the two velocities; in fact, it would be unscientific to make a different assumption (which would necessarily introduce greater complications) unless we were forced to it by unquestioned experimental fact. Accordingly, we shall make the assumption and shall state it as postulate R'' :

POSTULATE R'' . *The velocity of light in free space, measured on an unaccelerated system of reference S by means of units belonging to S , is independent of the absolute value of the velocity of S .*

POSTULATE R . The postulate obtained by combining R' and R'' will, for convenience, often be referred to as postulate R .

Now, since unaccelerated velocity is completely determined when the absolute value of the velocity and the direction of the motion are given the truth of the following theorem is an immediate consequence of theorem I. and postulate R'' .

THEOREM II. *The velocity of light in free space, measured on an unaccelerated system of reference S by means of units belonging to S , is independent of the velocity of S (MR).*

The second postulate of relativity has usually been stated in a form different from that given above in R' and R'' or R . In fact, the truth of theorem I. has often been taken as part of the *assumption* in this postulate, notwithstanding that I. can be derived from M and R' . Now, it is precisely the assumption of I. that has given most difficulty to some persons. It is believed that a part of this difficulty will disappear in view of the fact that I. is here *demonstrated* by means of M and R' .

For the sake of convenience in future discussion one of the customary formulations of the second postulate is appended here. It must be remembered, however, that it is not a separate constituent part of our present body of doctrine but is already contained in M and R , in part directly and in part as a necessary consequence of these postulates.

POSTULATE $\bar{R}$. *The velocity of light in free space, measured on an unaccelerated system of reference S by means of units belonging to S , is independent of the velocity of S and of the unaccelerated velocity of the light-source.*

From the very nature of the postulate R'' it is difficult to obtain direct experimental evidence for or against it. It seems, however, as we have

already pointed out, that one who accepts theorem I. can hardly refuse to assume R'' . But theorem I. is a logical consequence of postulates M and R' , as we have shown. Moreover, from what follows it will be seen that we have occasion to make no further assumptions which can in any way run counter to currently accepted notions. Consequently, it would seem that the experimental evidence for or against the whole theory of relativity must center around postulates M and R' . We have already given some account of the experimental evidence for these postulates. In connection with theorems to be derived later (in this paper and in another which the writer has in preparation) further reference will be given to the existing experimental evidence and some other possible lines of research in this direction will be pointed out.

It is generally conceded that the strange conclusions which flow from the theory of relativity are due to postulate R (or to postulate $\bar{R}$, in the customary formulation). In view of the theorem I. above and our discussion of its consequences, it is now clear¹ that the strangeness in the conclusions of relativity is due to that part of R which is contained in R' . It is important therefore to have a careful analysis of this postulate and especially to know alternative forms, which, in view of the other postulates, are logically equivalent to it. One such form has already been given by Tolman (l. c., p. 36), who has also urged the importance of the general problem. In the second paper of this series the alternative form due to Tolman will be subjected to a fresh analysis. As already pointed out in the Introduction, other alternative forms will also be given.

§ 4. *The Postulates V and L.*—It has been customary for writers on relativity to state explicitly only the postulates M and R . But every one, as a matter of fact, has made further assumptions concerning the relations of the two systems. These assumptions in some form are essential to the initial arguments and to the conclusions which are drawn by means of them. To the present writer it seems preferable to have these assumptions explicitly stated. Among the several forms, any one of which might be chosen, there is one which seems to us to be decidedly simpler than any of the others; and it is this one which we shall employ here. We state the postulates V and L as follows:

POSTULATE V . *If the velocity of a system of reference S_2 relative to a system of reference S_1 is measured by means of the units belonging to S_1 and if the velocity of S_1 relative to S_2 is measured by means of the units belonging to S_2 the two results will agree in absolute value.*

This velocity we shall call the relative velocity of the two systems. The direction line of this velocity will be called the line of relative motion of the two systems.

¹ This has already been pointed out by Tolman, l. c., pp. 27-28.

POSTULATE *L*. *If two systems of reference S_1 and S_2 move with unaccelerated relative velocity and if a line segment l is perpendicular to the line of relative motion of S_1 and S_2 and is fixed to one of these systems, then the length of l measured by means of the units belonging to S_1 will be the same as its length measured by means of the units belonging to S_2 .*

The essential content of these two postulates may be stated in simpler terms (but less accurately) if one allows the explicit introduction of the observer. Thus *V* is roughly equivalent to the following statement: *Two observers whose relative motion is uniform will agree in their measurement of that uniform relative motion.* As an approximate equivalent of *L* we have: *Two observers whose relative motion is uniform will agree in their measurement of length in a line perpendicular to their line of relative motion.*

It will be observed that these two postulates are nothing more than explicit statements of notions which underlie the classic theories of mechanics. The first is assumed in supposing that there exists such a thing as the relative motion of two bodies which are not at rest relatively to each other. The second is nothing more than the statement of a portion of the idea which lies at the bottom of our conception of such a thing as the length of a rod or other object.

Since these two postulates are universally accepted, the question might naturally arise, Why state them at all? Is it not enough simply to take them for granted? The answer is that there are other notions which have heretofore met with the same universal acceptance and which do not agree with the postulates of relativity. Therefore it seems to be desirable—in fact, to be essential to proper logical procedure—to state explicitly just those *assumptions* concerning the relation of the two systems of reference which we shall have occasion to employ in argument. Only in this way is one able to see exactly on what basis our strange conclusions rest.

We shall make a digression here to say one further word about postulate *L*. In part II. we shall draw the conclusion that length in the line of motion is not independent of the velocity with which the system is moving. In view of this the question arises as to why we must assume that length in a line perpendicular to the motion is independent of the motion. The answer is that we are under no such necessity, that we are at liberty to assume that length in a line perpendicular to the motion is dependent on the velocity of such motion. In fact, the general formulation of such an hypothesis has already been made by E. Riecke.¹ This hypothesis, however, is undoubtedly more complicated and less elegant than the one which we have made; and the latter, as we shall see, is in

¹ Göttinger Nachrichten, Math. Phys., 1911, pp. 271-277.

conflict with no known experimental facts. Therefore, following that instinct which has always wisely guided the physicist, we make the simplest hypothesis¹ which is in agreement with and explanatory of the totality of experimental facts at present known. If at any time experiments are set forth which do not agree with the theory developed on the basis of the above postulates, then will be the time to consider the question of introducing a more complicated postulate in place of our postulate *L* above.

§ 5. *Consistency and Independence of the Postulates.*—Throughout the paper it will be *assumed* that the postulates as stated are consistent; that is to say, no attempt will be made to prove their consistency. The fact that no contradictory conclusions have been drawn from the postulates will be accepted as (partial) evidence that they are mutually consistent. Moreover, from their very nature and from the differing range of applicability of the several postulates it is difficult to conceive how any one of them can possibly contradict conclusions which may be drawn from the others.

There is another question also which it is our purpose to pass over without discussion, namely, the question of the logical independence of the postulates. Is any postulate or a part of any postulate a logical consequence of the remaining postulates? This question is important from the point of view of formal logic, but in the present case its value to physical science is probably small.

§ 6. *Other Postulates Needed.*—From the postulates stated above it is possible to draw only those conclusions of the theory of relativity which are of a general nature. If, for instance, it is desired to study the nature of mass or the relation of mass and energy in this theory, it is necessary to have some assumption concerning mass in the first case and concerning both mass and energy in the second case. Thus we might assume the conservation laws of mass, energy, electricity, and momentum and deduce the joint consequences of these assumptions and those given above. It is our purpose to return to this matter in a future paper. For the present we are concerned only with the postulates above stated and their consequences.

II. RELATIVE MEASUREMENTS OF TIME AND SPACE IN TWO SYSTEMS OF REFERENCE. TRANSFORMATIONS.

§ 7. *Relations Between the Time Units.*—Let us consider three systems of reference *S*, *S*₁ and *S*₂ related to each other in the following manner: The lines of relative motion of *S* and *S*₁, of *S* and *S*₂, of *S*₁ and *S*₂ are all

¹ This hypothesis is in agreement with Einstein's theory of relativity.

parallel; S_1 and S_2 have a relative velocity v^1 ; S and S_1 have a relative velocity $\frac{1}{2}v$ in one sense and S and S_2 have a relative velocity $\frac{1}{2}v$ in the opposite sense. The system S consists of a single light-source, and this source is symmetrically placed with respect to two points of which one is fixed to S_1 and the other is fixed to S_2 . This is possible as a permanent relation on account of the relative motions of the three systems. For convenience, let us assume S to be at rest.

We shall now suppose that observers on the systems S_1 and S_2 measure the velocity of light as it emanates from the source S . Let a point A in S_1 and a point B in S_2 which are symmetrically placed with respect to the light-source S , move along the lines l_1 and l_2 ; these lines are parallel. From postulate L it follows that observers on S_1 and S_2 will obtain the same measurement of the distance between l_1 and l_2 . Denote this distance by d . On account of postulate M neither observer is able to detect his motion. Therefore he will make his observations on the assumption that his system is at rest; that is to say, his measurements will be made by means of the units belonging to his system and no corrections will be made on account of the motion of the system. Let the observer on S_1 reflect a beam of light SA from the point A to a point C on l_2 and back to A ; and let the observed time of passage of the light from A to C and back to A be t . Since the observer assumes his system to be at rest he will suppose that the ray of light passes (in both directions) along the line AC which is perpendicular to l_1 and l_2 . His measurement of the distance traversed by the ray of light in time t will therefore be $2d$. Hence he will obtain as a result

$$\frac{2d}{t} = c,$$

where c is his observed velocity of light.

Similarly, an observer on S_2 , supposing his system to be at rest, finds the time t_1 which it requires for a ray of light to pass from B to D and return, the ray employed being gotten by reflecting a ray SB at B . Thus the second observer obtains the result

$$\frac{2d}{t_1} = c_1,$$

where c_1 is his observed velocity of light.

¹ Note that postulate V is required to make this hypothesis legitimate.

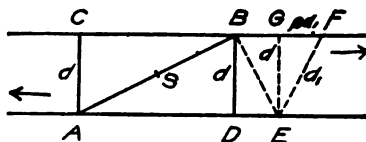


Fig. 1.

Now, from the assumed relations among the systems S , S_1 and S_2 and from the homogeneity of space it follows that the two observations which we have supposed to be made must lead to the same estimate for the velocity of light. This is readily seen from the fact that the observations were made in such a way that the effect due to either the absolute value or the direction of the motion of the systems S_1 and S_2 is the same in the two cases. In other words, if we denote by L_1 and L_2 the quantities measured on S_1 and S_2 respectively, then the relation of L_1 to S_1 is precisely the same as that of L_2 to S_2 ; and hence the numerical results are identical, as one sees from the definition of systems of reference. Therefore we have $c_1 = c$.

Let us now suppose that the observer at A is watching the experiment at B . To him it appears that B is moving with a velocity v ; we shall assume that the apparent motion is in the direction indicated by the arrow. To the observer at B it appears that the ray of light traverses BD from B to D and returns along the same line to B . To the observer at A it appears that the ray traverses the line BEF , F being the point which B has reached by the time that the ray has returned to the observer at this point. If EG is perpendicular to l_2 and d_1 is the length of EF as measured by means of units belonging to S_1 , then, evidently, GF (when measured in the same units) is βd_1 , where $\beta = v/\bar{c}$ and $\bar{c}$ is the (apparent) velocity of light as estimated in this case by the observer at A . From the right triangle EFG it follows at once that we have

$$d_1 = \frac{d}{\sqrt{1 - \beta^2}}.$$

Now, if l is the time which is required, according to the observer at A , for the light to traverse the path BEF , then we have

$$\frac{2d_1}{l} = \frac{2d}{l\sqrt{1 - \beta^2}} = \bar{c}.$$

So far in our argument in this section we have employed only those of our postulates which are generally accepted by both the friends and the foes of relativity. Now we come to the place where the men of the two camps must part company.

Let us introduce for the moment the following additional hypothesis.

ASSUMPTION A. *The two estimates c and $\bar{c}$ of the velocity of light obtained as above by the observer at A are equal.*

Now we have shown that c is equal to c_1 . Hence we may equate the values of c_1 and $\bar{c}$ given above; thus we have

$$\frac{2d}{l_1} = \frac{2d}{l\sqrt{1 - \beta^2}};$$

or

$$t_1 = l\sqrt{1 - \beta^2}.$$

But t_1 and l are measures of the same interval of time, t_1 being in units belonging to S_2 and l being in units belonging to S_1 . Hence to the observer on S_1 the ratio of his time unit to that of the system S_2 appears to be $\sqrt{1 - \beta^2} : 1$. On the other hand, it may be shown in exactly the same way that to the observer on S_2 the ratio of his time unit to that of the system S_1 appears to be $\sqrt{1 - \beta^2} : 1$. That is, the time units of the two systems are different and each observer comes to the same conclusion as to the relation which the unit of the other system bears to his own.

This important and striking result may be stated in the following theorem:

THEOREM III. *If two systems of reference S_1 and S_2 move with a relative velocity v and β is defined as the ratio of v to the velocity of light estimated in the manner indicated above, then to an observer on S_1 the time unit of S_1 appears to be in the ratio $\sqrt{1 - \beta^2} : 1$ to that of S_2 while to an observer on S_2 the time unit of S_2 appears to be in the ratio $\sqrt{1 - \beta^2} : 1$ to that of S_1 (MVLA).*

Let us now bring into play our postulate R' . In theorem I. we have already seen that a logical consequence of M and R' is that the velocity of light, as observed on a system of reference, is independent of the direction of motion of that system. Now, if c and $\bar{c}$ as estimated above differ at all, that difference can be due only to the direction of motion of S_1 , as one sees readily from postulate R' and the method of determining these quantities. Hence the statement which we made above as assumption A is a logical consequence of postulates M and R' . Therefore we are led to the following corollary of the above theorem:

COROLLARY. *Theorem III. may be stated as depending on (MVLR') instead of on (MVLA).*

Let us now go a step further and employ postulate R'' . From theorem I. and postulates R' and R'' it follows that the observed velocity of light is a pure constant for all admissible methods of observation. If we make use of this fact the preceding result may be stated in the following simpler form:

THEOREM IV. *If two systems of reference S_1 and S_2 move with a relative velocity v and β is the ratio of v to the velocity of light, then to an observer on S_1 the time unit of S_1 appears to be in the ratio $\sqrt{1 - \beta^2} : 1$ to that of S_2 while to an observer on S_2 the time unit of S_2 appears to be in the ratio $\sqrt{1 - \beta^2} : 1$ to that of S_1 (MVLR).*

Let us subject these remarkable results to a further analysis. Theorem III., its corollary and theorem IV. all agree in the extraordinary conclu-

sion that the time units of the two systems of reference S_1 and S_2 are of different lengths. Just how much they differ is a secondary matter; that they differ at all is the surprising and important thing. As postulates M , V , L are generally accepted and have not elsewhere led to such strange conclusions it is natural to suppose that the strangeness here is not due to them.

Referring to the argument carried out above, we see that no unusual conclusions were reached until we had introduced and made use of assumption A . Moreover we have seen that this assumption itself is a logical consequence of M and R' . Further, R'' is not involved either in theorem III. or in its corollary. But these already involve the strange features of our results. Hence the conclusion is irresistible that the extraordinary element in these results is due to postulate R' —or to speak more accurately, to just that part of it which it is necessary to use in connection with M in order to prove A as a theorem.

This result is important, as the following considerations show. Postulates V and L state laws which have been universally accepted in the classical mechanics. Postulate M is a direct generalization from experiment, and the generalization is legitimate according to the usual procedure of physicists in like situations. Postulate R' is the statement of a principle which has long been familiar in the theory of light and has met with wide acceptance. Thus we see that no one of these postulates, in itself, runs counter to currently accepted physical notions. And yet just these postulates alone are sufficient to enable us to conclude that corresponding time units in two systems of reference are of different magnitude. In the next section we shall show on the basis of the same postulates that the corresponding units of length in the two systems are also different. Thus the most remarkable elements in the conclusions of the theory of relativity are deducible from postulates M , V , L , R' alone; and yet these are either generalizations from experiment or statement of laws which have usually been accepted. Hence we conclude: *The theory of relativity, in its most characteristic elements, is a logical consequence of certain experiments together with certain laws which have for a long time been accepted.*

One other remark, of a totally different nature, should be made with reference to the characteristic result of theorem IV. It has to do with the relation between the time units of the two systems. This relation is intimately associated with the fact that each observer makes his measurements on the hypothesis that his own system is at rest, while the other system is moving past him with a velocity v . If both observers should agree to call S fixed and if further in this modified "universe" our

postulates V, L, R , were still valid it would turn out that the two observers would find their time units in agreement. But, in view of M , the choice of S as fixed would undoubtedly seem perfectly arbitrary to both observers; and the content of the modified postulate R would be essentially different from that of the postulate as we have employed it. Hence, if we accept R as it stands—or, indeed, even a certain part of it, as we have shown above—we must conclude that the time units in the two systems are not in agreement, in fact, that their ratio is that stated in the theorems above.

§ 8. *Relation Between the Units of Length.*—Let us consider three systems of reference S, S_1 and S_2 related in the same manner as in the preceding section except that now the two lines l_1 and l_2 coincide. We suppose that S_1 is moving in the direction indicated by the arrow at A and that S_2 is moving in the direction indicated by the arrow at B .

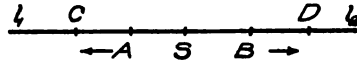


Fig. 2.

We suppose that observers at A and B again measure the velocity of light as it emanates from S , this time in the direction of the line of motion. Each will carry out his observations on the supposition that his system is at rest, for from M it follows that he cannot detect the motion of his system. The observer at A measures the time t_1 of passage of a ray of light from A to C and return to A , the length of AC being d when the measurement is made with a unit belonging to S_1 . Likewise, the observer at B measures the time t_2 of passage of a ray of light from B to D and return to B , the length of BD being d when measured with a unit belonging to S_2 .

Just as in the preceding case it may be shown that the two observers must obtain the same estimate for the velocity of light. But the estimate of the observer at A is $2d/t_1$ while that of the observer at B is $2d/t_2$. Hence

$$t_1 = t_2;$$

that is, the number of units of time required for the passage of the ray at A and of the ray at B is the same, the former being measured on S_1 and the latter on S_2 . Moreover, the measure of length is the same in the two cases. But the units of time, as we saw in the preceding section, do not have the same magnitude. Hence the units of length of the two systems along their line of motion do not have the same magnitude; and the ratio of units of length is the same as the ratio of units of time.

Combining this result with theorem III., its corollary, and theorem IV. we have the following three results:

THEOREM V. *If two systems of reference S_1 and S_2 move with a relative*

velocity v and β is defined as the ratio of v to the velocity of light estimated in the manner indicated in the first part of § 7, then to an observer on S_1 the unit of length of S_1 along the line of relative motion appears to be in the ratio $\sqrt{1 - \beta^2} : 1$ to that of S_2 while to an observer on S_2 the unit of length of S_2 along the line of relative motion appears to be in the ratio $\sqrt{1 - \beta^2} : 1$ to that of S_1 (MVLA).

COROLLARY. Theorem V. may be stated as depending on (MVL R') instead of on (MVLA).

THEOREM VI. If two systems of reference S_1 and S_2 move with a relative velocity v and if β is the ratio of v to the velocity of light, then to an observer on S_1 the unit of length of S_1 along the line of relative motion appears to be in the ratio $\sqrt{1 - \beta^2} : 1$ to that of S_2 while to an observer on S_2 the unit of length of S_2 along the line of relative motion appears to be in the ratio $\sqrt{1 - \beta^2} : 1$ to that of S_1 (MVL R).

We might make an analysis of these results similar to that which we gave for the corresponding results in the preceding section. But it would be largely a repetition. It is sufficient to point out that the remarkable conclusions as to units of length in the two systems rest on just those assumptions which led to the strange results as to the units of time.

§ 9. *Simultaneity of Events Happening at Different Places.*—Let us now assume two systems of reference S and S' moving with a uniform relative velocity v . Let an observer on S' undertake to adjust two clocks at different places so that they shall simultaneously indicate the same time. We will suppose that he does this in the following very natural manner:¹ Two stations A and B are chosen in the line of relative motion of S and S' and at a distance d apart. The point C midway between these two stations is found by measurement. The observer is himself stationed at C and has assistants at A and B . A single light signal is flashed from C to A and to B , and as soon as the light ray reaches each station the clock there is set at an hour agreed upon beforehand. The observer on S' now concludes that his two clocks, the one at A and the other at B , are simultaneously marking the same hour; for, in his opinion (since

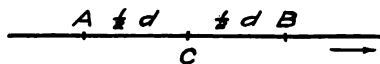


Fig. 3.

he supposes his system to be at rest), the light has taken exactly the same time to travel from C to A as to travel from C to B .

Now let us suppose that an observer on the system S has watched the work of regulating these clocks on S' . The distances CA and CB appear to him to be

¹ Compare Comstock, *Science*, N. S., 31 (1900): 767-772.

$$\frac{1}{2}d\sqrt{1-\beta^2}$$

instead of $\frac{1}{2}d$. Moreover, since the velocity of light is independent of the velocity of the source, it appears to him that the light ray proceeding from C to A has approached A at the velocity $c + v$, where c is the velocity of light, while the light ray going from C to B has approached B at the velocity $c - v$. Thus to him it appears that the light has taken longer to go from C to B than from C to A by the amount

$$\frac{\frac{1}{2}d\sqrt{1-\beta^2}}{c-v} - \frac{\frac{1}{2}d\sqrt{1-\beta^2}}{c+v} = \frac{vd\sqrt{1-\beta^2}}{c^2-v^2}.$$

But since $\beta = v/c$ the last expression is readily found to be equal to

$$\frac{v}{c^2} \cdot \frac{d}{\sqrt{1-\beta^2}}.$$

Therefore, to an observer on S the clocks on S' appear to mark different times; and the difference is that given by the last expression above.

Thus we have the following conclusion:

THEOREM VII. *Let two systems of reference S and S' have a uniform relative velocity v . Let an observer on S' place two clocks at a distance d apart in the line of relative motion of S and S' and adjust them so that they appear to him to mark simultaneously the same time. Then to an observer on S the clock on S' which is forward in point of motion appears to be behind in point of time by the amount*

$$\frac{v}{c^2} \cdot \frac{d}{\sqrt{1-\beta^2}},$$

where c is the velocity of light and $\beta = v/c$ (MVL R).

It should be emphasized that the clocks on S' are in agreement in the only sense in which they can be in agreement for an observer on that system who supposes (as he naturally will) that his own system is at rest—notwithstanding the fact that to an observer on the other system there appears to be an irreconcilable disagreement depending for its amount directly on the distance apart of the two clocks.

According to the result of the last theorem the notion of simultaneity of events happening at different places is indefinite in meaning until some convention is adopted as to how simultaneity is to be determined. In other words, *there is no such thing as the absolute simultaneity of events happening at different places.*

§ 10. *Transformation of Space and Time Coördinates.*—It is now an easy matter to derive the Einstein formulæ¹ for the transformation of space

¹ Jahrbuch der Radioaktivität, 4 (1907): 418-420.

and time coördinates. Let two systems of reference S and S' have the relative velocity v in the line l . Let systems of rectangular coördinates be attached to the systems of reference S and S' in such way that the x -axis of each system is in the line l , and let the y -axis and the z -axis of one system be parallel to the y -axis and the z -axis respectively of the other system. Let the origins of the two systems coincide at the time $t = 0$. Furthermore, for the sake of distinction, denote the coördinates on S by x, y, z, t and those on S' by x', y', z', t' . We require to find the value of the latter coördinates in terms of the former.

From postulate L it follows at once that $y' = y$ and $z' = z$. Let an observer on S consider a point which at the time $t = 0$ appears to him to be at distance¹ x from the $y'z'$ -plane; at time $t = t$ it will appear to him to be at the distance $x - vt$ from the $y'z'$ -plane. Now, by an observer on S' this distance is denoted by x' . Then from theorem, VI. we have

$$x' \sqrt{1 - \beta^2} = x - vt.$$

Now consider a point at the distance x from the yz -plane at time $t = t$ in units of system S . From theorem VII. it follows that to an observer on S the clock on S' at the same distance x from the yz -plane will appear behind by the amount

$$\frac{v}{c^2} x,$$

where c is the velocity of light. That is, in units of S this clock would register the time

$$t - \frac{v}{c^2} x.$$

Hence, by means of theorem IV., we have at once the result

$$t' \sqrt{1 - \beta^2} = t - \frac{v}{c^2} x.$$

Solving the two equations involving x' and t' and collecting results, we have

$$\begin{aligned} t' &= \frac{1}{\sqrt{1 - \beta^2}} \left(t - \frac{v}{c^2} x \right), \\ (A) \quad x' &= \frac{1}{\sqrt{1 - \beta^2}} (x - vt), & (MVLR) \\ y' &= y, \\ z' &= z, \end{aligned}$$

where $\beta = v/c$ and c is the velocity of light.

In the same way we may obtain the equations which express t, x, y, z

¹ The algebraic sign of the distance is supposed to be taken into account in the value of x .

in terms of t', x', y', z' . But these can be found more easily by solving equations (A) for t', x', y', z' . Thus we have

$$\begin{aligned} t &= \frac{1}{\sqrt{1 - \beta^2}} \left(t' + \frac{v}{c^2} x' \right), \\ (A_1) \quad x &= \frac{1}{\sqrt{1 - \beta^2}} (x' + vt'), \\ y &= y' \\ z &= z' \end{aligned} \quad (MVL R)$$

These two sets of equations (A) and (A₁) are identical in form except for the sign of v . This symmetry in the transformations constitutes one of their chief points of interest.

Our method of proof of these formulæ is very different from that of Einstein, as a comparison will readily show. The difference is due primarily to our use of postulates *V* and *L* instead of the assumptions of Einstein.¹

In a paper² entitled "The Common Sense of Relativity" Campbell has made some interesting remarks concerning these transformations.

§ 11. *The Addition of Velocities*.—For the sake of completeness³ in the presentation of the fundamental results of relativity and for use in the next section we derive here the formulæ for addition of velocities due to Einstein.⁴

Let the velocity of a point in motion be represented in units belonging to S' and to S by means of the equations

$$\begin{aligned} x' &= u_x t', \quad y' = u_y t', \quad z' = u_z t'; \\ x &= u_x t, \quad y = u_y t, \quad z = u_z t, \end{aligned}$$

respectively. In the first of these substitute for t', x', y', z' their values given by (A), solve for $x/t, y/t, z/t$ and replace these quantities by their equals u_x, u_y, u_z , respectively. Thus we have

$$\begin{aligned} u_x &= \frac{u_{x'} + v}{1 + \frac{vu_{x'}}{c^2}}, \\ (B) \quad u_y &= \frac{\sqrt{1 - \beta^2}}{1 + \frac{vu_{x'}}{c^2}} u_{y'}, \\ u_z &= \frac{\sqrt{1 - \beta^2}}{1 + \frac{vu_{x'}}{c^2}} u_{z'}. \end{aligned} \quad (MVL R)$$

¹ Einstein, l. c., p. 420, footnote.

² Phil. Mag., 21 (1911): 502-517; see esp. pp. 505-507.

³ See remarks in the Introduction.

⁴ Einstein, l. c., pp. 422-424.

From these results it follows that the law of the parallelogram of velocities is only approximate. This conclusion of the theory of relativity has given rise, in the minds of some persons, to the most serious objections to the entire theory.

Suppose that both the velocities considered above are in the line of relative motion of S and S' . Then we have

$$u = \frac{v + u'}{1 + \frac{vu'}{c^2}}.$$

This equation gives rise to the following theorem:

THEOREM VIII. *If two velocities, each of which is less than c , are combined the resultant velocity is also less than c (MVLRL).*

To prove this we substitute in the preceding equation for v and u' the values

$$v = c - k, \quad u' = c - l,$$

where each of the numbers k and l is positive and less than c . Then the equation becomes

$$u = c \frac{2c - k - l}{2c - k - l + \frac{kl}{c}}.$$

The second member is evidently less than c . Hence the theorem.

If, however, either one (or both) of the velocities v and u' is equal to c —and hence k or l (or both) is equal to zero—we see at once from the last equation that $u = c$. Hence, we have the following result:

THEOREM IX. *If a velocity c is compounded with a velocity equal to or less than c , the resultant velocity is c (MVLRL).*

Remark.—A conclusion of importance is implicitly involved in the results obtained in §§ 7–11. It can probably be seen in the simplest way by reference to the first two equations (A), these being nothing more nor less than an analytic formulation of theorems IV. and VI. If β is in absolute value greater than 1—whence $1 - \beta^2$ is negative—the transformation of time coördinates from one system to the other gives an imaginary result for the time in one system if the time in the other system is real. Likewise, measurement of length in the direction of motion is imaginary in one system if it is real in the other. Both of these conclusions are absurd and hence the absolute value of β is equal to or less than 1. If it is one, then any length in one system, however short, would be measured in the other as infinite; and a like result holds for time. Hence β is less than 1. But $\beta = v/c$, the ratio of the relative velocity of the two systems to that of light. Hence, the velocity of light is a maxi-

num which the relative velocity of two systems may approach but can never reach. This may be formulated in the following theorem:

THEOREM X. *The velocity of light is a maximum which the velocity of a material system may approach but can never reach (MVL R).*

It should be pointed out that this theorem may also be proved directly from theorem IX, as one can readily show. This fact will be useful in the next section.

§12. *Logical Equivalents of the Postulates.*—We shall now show that theorem IX. is in a certain sense a logical equivalent of *R*. From IX. it follows, as we have seen in theorem X., that the velocity of a material body is less than the velocity of light. But the source of light is always a material body; and therefore no light source can have a velocity as great as that of light. Now, the following is a natural hypothesis:

POSTULATE B. *The velocity of the light source cannot add¹ to the velocity of light a greater velocity than that of the source itself. Likewise, the velocity of a system of reference cannot add to the velocity of light a greater velocity than that of the source itself.*

Now, from theorem IX. it follows that if any velocity less than that of light is compounded with that of light the resultant is the velocity of light. Hence, if we assume theorem IX. and postulate *B* we can conclude as a consequence postulates *R'* and *R''*. Hence we have the following result:

THEOREM XI. *Postulates (MVL B) and theorem IX. are a logical equivalent of postulates (MVL R).*

That is, in our system of postulates *R* may be replaced by theorem IX. and postulate *B*, and the resulting total body of postulates and theorems will be unaltered.

Now theorem IX. was proved by means of formulæ (*B*) alone; and formulæ (*B*) are a direct consequence of formulæ (*A*) above. Hence postulate *R* may be proved solely from postulate *B* and formulæ (*A*) if these are assumed to be true. Further, the third and fourth equations in (*A*) are equivalent to postulate *L*. We shall show that a special case of the first two formulæ (*A*) is postulate *V*. Putting $x' = 0$ we have $x/t = v$. That is, to an observer on *S* the system *S'* appears to move with the velocity *v*. Now the first two equations in (*A*₁) may be obtained algebraically by solving the first two in (*A*). Then in the second equation of (*A*₁) put $x = 0$; thus we have $x'/t' = -v$. That is, to an observer on *S'* the system *S* appears to move with the velocity $-v$. These two results together constitute our postulate *V*. Combining the several conclusions thus reached we have the following theorem:

¹ Addition is defined by saying that the sum of two velocities is the result of compounding them.

THEOREM XII. *Postulates (MB) and formulæ (A) are a logical equivalent of postulates (MVL_R).*

An analysis of the proof of formulæ (A) will show that they follow directly from theorems IV., VI., VII. and postulate V. Hence we have the following theorem as a corollary of the preceding:

THEOREM XIII. *Postulates (MVB) and theorems IV., VI., VII. are a logical equivalent of postulates (MVL_R).*

INDIANA UNIVERSITY,
June, 1912.

ON THE MOTION OF A DISC WITH THREE SUPPORTS ON A ROUGH PLANE.

BY PETER FIELD.

INTRODUCTION.

THE motion of a heavy body on a smooth horizontal plane was first studied by Poisson.¹ He supposed the body to have an apex such as a top or else to be bounded by a continuous surface which is in contact with the plane at some point.

In the fifth and eighth volumes of Crelle's *Journal für die reine und angewandte Mathematik* are three papers along the same line by Cournot.² At the beginning of his first article Cournot states that by following out reasoning analogous to that used by Poisson there is no difficulty in writing the equations of motion for a body which has three points of contact with a rough plane, but that the integrations can only be performed by laborious approximations and consequently their interest arises only in connection with their applications. Since that time articles and text books³ have been written in which problems dealing with the motion of a rigid body on a plane have been studied. Within the last fifteen years Painlevé's "*Leçons sur le frottement*" has apparently inspired a number of articles which have concerned themselves with Coulomb's laws of friction. To this class belongs for instance the work by A. Mayer.⁴

It seems, however, that no effort has been made to write out and study the equations of motion for the case mentioned by Cournot. Although it is true that the equations are rather complicated, they do furnish approximate solutions for certain cases without difficulty and for that reason it seems worth while to give the problem some thought.⁵

¹ *Traité de Mécanique*.

² (a) *Mémoire sur le mouvement d'un corps rigide, soutenu par un plan fixe*. (b) *Du mouvement d'un corps sur un plan fixe, quand on a égard à la résistance du frottement, et qu' on ne suppose qu' un seul point de contact*. (c) *Du mouvement d'un corps sur un plan fixe, quand on a égard à la résistance du frottement*.

³ As for instance Routh, *Advanced Rigid Dynamics*; Appell, *Traité de Mécanique Rationnelle*; Klein und Sommerfeld, *Theorie des Kreisels*; Jellett, *Theory of Friction*, and numerous others.

⁴ *Zur Theorie der gleitenden Reibung*, Leipzig Ber. 53, pp. 235-318. For additional references along this line see Appell, *Mécanique*, Vol. II., 3d edition, p. 127.

⁵ An interesting problem in plane motion, slightly related to the present one, is treated

STATEMENT OF THE PROBLEM AND GEOMETRICAL DISCUSSION.

Suppose there is given a disc which is supported on a rough plane by three short legs. It is proposed to study the motion (a) when the disc is initially given a large angular velocity ω_0 about an axis perpendicular to its plane and passing through its centroid, (b) when the disc is initially given a velocity v_0 of translation only.

For convenience the triangle 1, 2, 3 (Fig. 1) formed by the three points of support will be taken isosceles with 23 as base and the centroid G will be placed on the line of symmetry. The coefficient of friction at the point 1 will be taken as μ_1 while at 2 and 3 it will be taken as μ_2 . The reactions at the points of support being F_1 , F_2 , and F_3 , the frictional forces are $\mu_1 F_1$, $\mu_2 F_2$, $\mu_2 F_3$. The direction of the acceleration of the

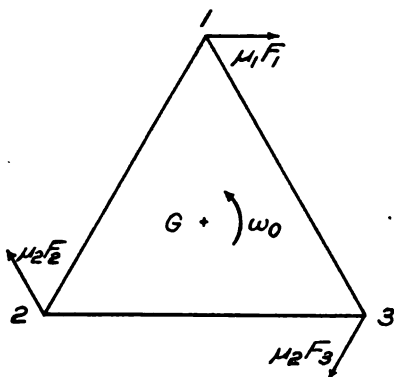


Fig. 1.

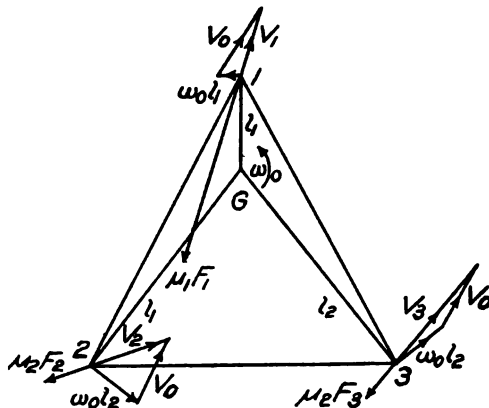


Fig. 2.

centroid is coincident with that of the resultant R of the frictional forces and its magnitude is proportional to this resultant. In order to have a large acceleration it is therefore necessary to make the resultant of the frictional forces as large as possible. This can be accomplished in various ways; for instance by placing the centroid nearer to 1 than to the other two points or by taking μ_1 much larger than μ_2 . In case 1 2 3 is an equilateral triangle with G at its center and $\mu_1 = \mu_2$, the resultant of the frictional forces is zero and the centroid remains stationary. The values of F_1 , F_2 , and F_3 are not exactly the same as for a statical problem but it will appear later that the difference is slight when the supports are short.

An idea of the effect of combining a velocity of translation, v_0 , with the velocity of rotation, ω_0 is shown in Fig. 2. The velocities of the supports are represented by v_1 , v_2 , v_3 . Each of these velocities is the by Brill, *Vorlesungen zur Einführung in die Mechanik raumfüllender Massen*, p. 31. The problem is introduced to furnish an example of a non-holonomic condition.

resultant of two velocities, one due to the rotation, which is different for the different points, and the other v_0 , due to the translation, which is common.

For $\omega_0 = 0$ or a quantity so small that the velocity of the supports is practically the same as v_0 , the frictional forces are parallel and opposed in sense to v_0 , as in Fig. 3. The motion of the centroid will be uniformly retarded motion while the angular acceleration will be dependent on the moment of the three frictional forces about the centroid.

THE EQUATIONS OF MOTION.

In Fig. 4, 1 2 3 is the triangle formed by the three points of support. The position of the disc is determined when the coördinates (x, y) of the centroid and the angle θ which the line l_1 makes with the X axis are given.

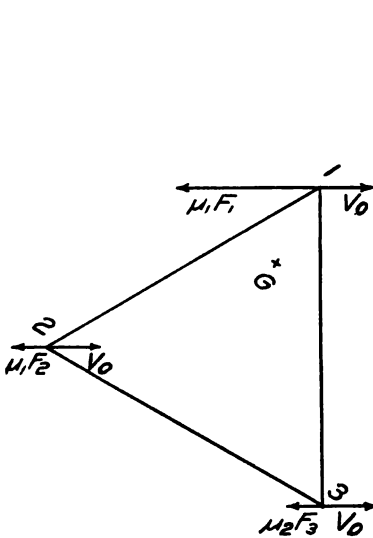


Fig. 3.

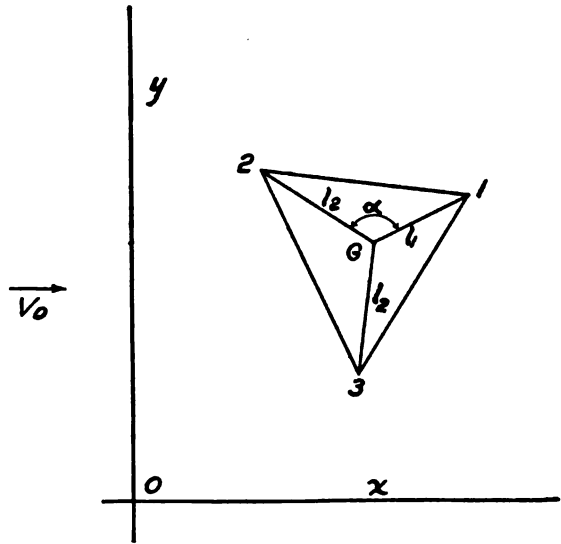


Fig. 4.

Let (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) be the coördinates of the points 1, 2, and 3, then

$$\begin{aligned} x_1 &= x + l_1 \cos \theta, & \dot{x}_1 &= \dot{x} - l_1 \dot{\theta} \sin \theta, \\ y_1 &= y + l_1 \sin \theta, & \dot{y}_1 &= \dot{y} + l_1 \dot{\theta} \cos \theta, \\ x_2 &= x + l_2 \cos (\theta + \alpha), & \dot{x}_2 &= \dot{x} - l_2 \dot{\theta} \sin (\theta + \alpha), \\ y_2 &= y + l_2 \sin (\theta + \alpha), & \dot{y}_2 &= \dot{y} + l_2 \dot{\theta} \cos (\theta + \alpha), \\ x_3 &= x + l_3 \cos (\theta - \alpha), & \dot{x}_3 &= \dot{x} - l_3 \dot{\theta} \sin (\theta - \alpha), \\ y_3 &= y + l_3 \sin (\theta - \alpha), & \dot{y}_3 &= \dot{y} + l_3 \dot{\theta} \cos (\theta - \alpha). \end{aligned}$$

Inasmuch as the frictional force at any point is opposite to the direction

in which the point is moving the equations of motion of the centroid are

$$m\ddot{x} = -\mu_1 F_1 \frac{\dot{x} - l_1 \dot{\theta} \sin \theta}{v_1} - \mu_2 F_2 \frac{\dot{x} - l_2 \dot{\theta} \sin (\theta + \alpha)}{v_2} - \mu_3 F_3 \frac{\dot{x} - l_2 \dot{\theta} \sin (\theta - \alpha)}{v_3}, \quad (1)$$

$$m\ddot{y} = -\mu_1 F_1 \frac{\dot{y} + l_1 \dot{\theta} \cos \theta}{v_1} - \mu_2 F_2 \frac{\dot{y} + l_2 \dot{\theta} \cos (\theta + \alpha)}{v_2} - \mu_3 F_3 \frac{\dot{y} + l_2 \dot{\theta} \cos (\theta - \alpha)}{v_3}. \quad (2)$$

Taking I as the moment of inertia of the disc about the centroid, the equation of the motion about the centroid is

$$\begin{aligned} I\ddot{\theta} = & -l_1 \mu_1 F_1 \cos \theta \frac{\dot{y} + l_1 \dot{\theta} \cos \theta}{v_1} + l_1 \mu_1 F_1 \sin \theta \frac{\dot{x} - l_1 \dot{\theta} \sin \theta}{v_1} \\ & - l_2 \mu_2 F_2 \cos (\theta + \alpha) \frac{\dot{y} + l_2 \dot{\theta} \cos (\theta + \alpha)}{v_2} + l_2 \mu_2 F_2 \sin (\theta + \alpha) \frac{\dot{x} - l_2 \dot{\theta} \sin (\theta + \alpha)}{v_2} \\ & - l_2 \mu_3 F_3 \cos (\theta - \alpha) \frac{\dot{y} + l_2 \dot{\theta} \cos (\theta - \alpha)}{v_3} + l_2 \mu_3 F_3 \sin (\theta - \alpha) \frac{\dot{x} - l_2 \dot{\theta} \sin (\theta - \alpha)}{v_3}. \end{aligned} \quad (3)$$

If d represents the common length of the three supports, F_1 , F_2 , and F_3 can be determined from the three following equations.

$$F_1 + F_2 + F_3 = mg, \quad (4)$$

$$\begin{aligned} F_2 l_2 \sin \alpha - F_3 l_2 \sin \alpha + d \mu_1 F_1 \frac{\dot{x} \sin \theta - \dot{y} \cos \theta - l_1 \dot{\theta}}{v_1} \\ + d \mu_2 \left(\frac{F_2}{v_2} + \frac{F_3}{v_3} \right) (\dot{x} \sin \theta - \dot{y} \cos \theta - l_2 \dot{\theta} \cos \alpha) = 0, \end{aligned} \quad (5)$$

$$\begin{aligned} -F_1 l_1 - F_2 l_2 \cos \alpha - F_3 l_2 \cos \alpha + d \mu_1 F_1 \frac{\dot{x} \cos \theta + \dot{y} \sin \theta}{v_1} \\ + d \mu_2 F_2 \frac{\dot{x} \cos \theta + \dot{y} \sin \theta - l_2 \dot{\theta} \sin \alpha}{v_2} + d \mu_3 F_3 \frac{\dot{x} \cos \theta + \dot{y} \sin \theta + l_2 \dot{\theta} \sin \alpha}{v_3} = 0, \end{aligned} \quad (6)$$

Equation (4) expresses the condition that the centroid remain at a constant distance from the xy -plane while (5) and (6), obtained by taking moments about l_1 and a line at right angles to l_1 in the plane of the disc, express the condition that the axis of angular momentum remain perpendicular to this plane. The equations show that F_1 , F_2 , F_3 depend both on the velocity of the centroid and on the length of the supports.

As a special case suppose $\dot{x}$ and $\dot{y}$ are so small that they can be neglected. This gives

$$F_1 = \frac{(l_2^2 + d^2 \mu_2^2) \cos \alpha}{(l_2^2 + d^2 \mu_2^2) \cos \alpha - (l_1 l_2 + d^2 \mu_1 \mu_2)} mg, \quad (7)$$

$$F_2 = \frac{1}{2} \frac{(\mu_1 l_2 - l_1 \mu_2) d \cot \alpha - l_1 l_2 - d^2 \mu_1 \mu_2}{(l_2^2 + d^2 \mu_2^2) \cos \alpha - (l_1 l_2 + d^2 \mu_1 \mu_2)} mg, \quad (8)$$

$$F_3 = \frac{1}{2} \frac{(\mu_1 l_2 - l_1 \mu_2) d \cot \alpha - l_1 l_2 - d^2 \mu_1 \mu_2}{(l_2^2 + d^2 \mu_2^2) \cos \alpha - (l_1 l_2 + d^2 \mu_1 \mu_2)} mg, \quad (9)$$

which are constant. From the nature of the problem our data are to be so chosen that F_1 , F_2 , and F_3 are positive quantities: as $\pi \geq \alpha \geq \pi/2$, this will be the case if $|(\mu_1 l_2 - l_1 \mu_2) d \cot \alpha| < l_1 l_2 + d^2 \mu_1 \mu_2$.

As a second special case suppose the velocity of rotation is so small that it can be neglected. Taking the velocity of the centroid along the X axis, $\dot{x} = v_1 = v_2 = v_3$, $\dot{y} = 0$. The values for F_1 , F_2 , F_3 then become

$$\begin{aligned} F_1 &= \frac{\mu_2 d \cos \theta - l_2 \cos \alpha}{(\mu_2 - \mu_1) d \cos \theta + l_1 - l_2 \cos \alpha} mg, \\ F_2 &= \frac{l_1 l_2 \sin \alpha - d l_1 \mu_2 \sin \theta + l_2 \mu_1 d \sin (\theta - \alpha)}{2 l_2 \sin \alpha [(\mu_2 - \mu_1) d \cos \theta + l_1 - l_2 \cos \alpha]} mg, \\ F_3 &= \frac{l_1 l_2 \sin \alpha + l_1 \mu_2 d \sin \theta - \mu_1 d l_2 \sin (\theta + \alpha)}{2 l_2 \sin \alpha [(\mu_2 - \mu_1) d \cos \theta + l_1 - l_2 \cos \alpha]} mg. \end{aligned}$$

In this case F_1 , F_2 , F_3 are functions of θ and the data would therefore need to be so chosen that the reactions are positive no matter what the value of θ .

As the general expressions for F_1 , F_2 , and F_3 are rather complicated we at this point simplify the problem by assuming that d is so small that the terms in equations (4), (5), (6) which contain d may be omitted. The values for the normal components of the reaction of the plane are then the same as for a statical problem:

$$\begin{aligned} F_1 &= \frac{l_2 \cos \alpha}{l_2 \cos \alpha - l_1} mg, \\ F_2 = F_3 &= \frac{l_1}{2(l_1 - l_2 \cos \alpha)} mg. \end{aligned}$$

These values of F_1 , F_2 , F_3 substituted in equations (1), (2), (3) define the motion of a disc with short supports on a rough horizontal plane.

SOLUTION OF THE EQUATIONS OF MOTION IN CASE THE BODY IS INITIALLY GIVEN A LARGE ANGULAR VELOCITY ω_0 ABOUT THE CENTROID.

Take the axes of reference so that the initial conditions are $t = 0$, $\dot{x} = 0$, $\dot{y} = 0$, $\dot{\theta} = \omega_0$, $x = 0$, $y = 0$, $\theta = 0$. As long as the velocity of the centroid is so small that $\sqrt{x^2 + y^2}$ can be neglected in comparison with $l_1 \dot{\theta}$ and $l_2 \dot{\theta}$, the equations (1), (2), (3) reduce to

$$\ddot{x} = \frac{\mu_1 l_2 - \mu_2 l_1}{l_2 \cos \alpha - l_1} g \cos \alpha \sin \theta = k \sin \theta, \quad (10)$$

$$\ddot{y} = - \frac{\mu_1 l_2 - \mu_2 l_1}{l_2 \cos \alpha - l_1} g \cos \alpha \cos \theta = -k \cos \theta, \quad (11)$$

$$\ddot{\theta} = \frac{\mu_2 - \mu_1 \cos \alpha}{l_2 \cos \alpha - l_1} \frac{mg l_1 l_2}{I} = -a. \quad (12)$$

The last equation gives

$$\theta = \omega_0 t - \frac{1}{2} a t^2.$$

Hence

$$\ddot{x} = k \sin(\omega_0 t - \frac{1}{2} a t^2), \quad \ddot{y} = -k \cos(\omega_0 t - \frac{1}{2} a t^2). \quad (13)$$

For a large ω_0 and a short interval of time this gives as a first approximation

$$\begin{aligned} \ddot{x} &= k \sin \omega_0 t, \quad \ddot{y} = -k \cos \omega_0 t, \\ x &= \frac{k}{\omega_0^2} (\omega_0 t - \sin \omega_0 t), \quad y = -\frac{k}{\omega_0^2} (1 - \cos \omega_0 t). \end{aligned} \quad (14)$$

These equations represent a cycloid, the radius of the generating circle being k/ω_0^2 . The curve will lie in the fourth or second quadrant according as k is positive or negative.

A somewhat better approximation is obtained by writing equations (13) in the form

$$\begin{aligned} \ddot{x} &= k \sin \omega_0 t \cos \frac{1}{2} a t^2 - k \cos \omega_0 t \sin \frac{1}{2} a t^2, \\ \ddot{y} &= -k \cos \omega_0 t \cos \frac{1}{2} a t^2 - k \sin \omega_0 t \sin \frac{1}{2} a t^2. \end{aligned}$$

Being $\frac{1}{2} a t^2$ is small we replace these equations by

$$\begin{aligned} \ddot{x} &= k \sin \omega_0 t - k \frac{1}{2} a t^2 \cos \omega_0 t, \\ \ddot{y} &= -k \cos \omega_0 t - k \frac{1}{2} a t^2 \sin \omega_0 t. \end{aligned}$$

The equations for the path of the centroid will then be

$$\begin{aligned} x &= \frac{k}{\omega_0^2} (\omega_0 t - \sin \omega_0 t) + \frac{3ka}{\omega_0^4} (1 - \cos \omega_0 t) \\ &\quad - \frac{ka}{\omega_0^4} \left[2\omega_0 t \sin \omega_0 t - \frac{1}{2} \omega_0^2 t^2 \cos \omega_0 t \right], \\ y &= -\frac{k}{\omega_0^2} (1 - \cos \omega_0 t) + \frac{ka}{\omega_0^4} t \omega_0 (1 + 2 \cos \omega_0 t) \\ &\quad - \frac{ka}{\omega_0^4} \sin \omega_0 t \left[3 - \frac{1}{2} \omega_0^2 t^2 \right]. \end{aligned}$$

It is convenient to think of these last equations as representing a curve whose form can be obtained from the cycloid determined by equations (14) by a slight deformation. The difference between the values of x and y in the two cases, for a given value of $\omega_0 t$, can then be considered as a correction. These corrections are shown for a few values of $\omega_0 t$ in the following table. The value of π is taken as 3.14.

Value of $\omega_0 t$.	x correction.	y correction.
$\frac{\pi}{2}$	$\frac{ka}{\omega_0^4} (-.14)$	$\frac{ka}{\omega_0^4} (-.2)$
π	$\frac{ka}{\omega_0^4} (1.07)$	$\frac{ka}{\omega_0^4} (-3.14)$
$\frac{3}{2}\pi$	$\frac{ka}{\omega_0^4} (12.42)$	$\frac{ka}{\omega_0^4} (-3.38)$
2π	$\frac{ka}{\omega_0^4} (19.72)$	$\frac{ka}{\omega_0^4} (18.84)$

SOLUTION OF THE EQUATIONS OF MOTION IN CASE θ IS SO SMALL THAT

$l_1\dot{\theta}$ AND $l_2\dot{\theta}$ CAN BE NEGLECTED IN COMPARISON WITH $\sqrt{\dot{x}^2 + \dot{y}^2}$.

Let the initial velocity of the centroid be along the X axis. The initial conditions are then $t = 0$, $\dot{x} = \dot{x}_0$, $\dot{y} = 0$, $v_1 = v_2 = v_3 = \dot{x}_0$. Equations (1), (2), (3) become

$$\ddot{x} = -\frac{\mu_1 F_1 + 2\mu_2 F_2}{m} = -\frac{(\mu_1 l_2 \cos \alpha - \mu_2 l_1)g}{l_2 \cos \alpha - l_1} = -j, \quad (15)$$

$$\ddot{y} = 0, \quad (16)$$

$$\begin{aligned} \ddot{\theta} &= \frac{(l_1 \mu_1 F_1 + 2l_2 \mu_2 F_2 \cos \alpha) \sin \theta}{I} \\ &= \frac{l_1 l_2 (\mu_1 - \mu_2) \cos \alpha mg}{l_2 \cos \alpha - l_1} \frac{1}{I} \sin \theta = -b \sin \theta. \end{aligned} \quad (17)$$

Equations (15) and (16) give for the motion of the centroid

$$x = \dot{x}_0 t - \frac{1}{2} j t^2, \quad y = 0.$$

From (17) it follows that the motion relative to the centroid is the same as that of a pendulum.

THE CASE OF TWO COINCIDENT SUPPORTS.

Two of the supports may be made coincident by taking $\alpha = \pi$. The triangle formed by the points of support would also degenerate into a straight line in case $\alpha = \pi/2$ and $l_1 = 0$, but this gives indeterminate values for the reactions of the supports.

For $\alpha = \pi$,

$$F_1 = \frac{l_2}{l_1 + l_2} mg,$$

and the reaction at the second point is

$$F_2 + F_3 = \frac{l_1}{l_1 + l_2} mg.$$

The equations (10), (11), (12) which determine the motion for a large spin and a small velocity of translation, become

$$\begin{aligned}\ddot{x} &= \frac{\mu_1 l_2 - \mu_2 l_1}{l_2 + l_1} g \sin \theta, \\ \ddot{y} &= - \frac{\mu_1 l_2 - \mu_2 l_1}{l_2 + l_1} g \cos \theta, \\ \ddot{\theta} &= - \frac{\mu_2 + \mu_1}{l_2 + l_1} \frac{mg}{I} l_1 l_2\end{aligned}$$

For a small spin and a large velocity of translation it follows from the equations (15), (16), (17) that the motion is given by

$$\begin{aligned}\ddot{x} &= - \frac{\mu_1 l_2 + \mu_2 l_1}{l_2 + l_1} g = -j, \\ \ddot{y} &= 0, \\ \ddot{\theta} &= \frac{l_1 l_2 (\mu_1 - \mu_2)}{l_2 + l_1} \frac{mg}{I} \sin \theta.\end{aligned}$$

It is apparent from the above equations that although the constants are somewhat simplified, the motion is of the same general nature as before.

UNIVERSITY OF MICHIGAN,
June, 1912.

THE EFFECT OF VIBRATION ON THE RESISTANCE OF METALS.

BY H. L. BRAKEL.

THE effect of tension, torsion and hydrostatic pressure on the resistance of metals has been the subject of investigations extending over a considerable period. The more recent work along these lines has been done by the following: G. Ercolini¹ has investigated the effect of tension and torsion on the resistance of nickel, N. F. Smith² has done the same for iron, brass, and copper under quite different conditions. W. E. Williams³ has subjected lead, aluminum, manganin, and bismuth to hydrostatic pressure and has measured the changes produced in the resistance. He has also studied the effect of tension on the resistance of bismuth. P. W. Bridgeman⁴ has subjected manganin wires to very high hydrostatic pressures and has found that the pressure-resistance coefficient is a linear function of the pressure for a very wide range of pressures.

The work here described is a study of the effect of vibration on the resistance of metals. For this purpose it was essential, first of all, to devise a method by which metals, in the form of wires, could be vibrated with a constant amplitude. For a thorough study it was necessary to be able to change the amplitude through a wide range and still keep it constant for each range, and in order to save time the period of vibration should be controllable independent of other conditions. The following device proved highly satisfactory.

APPARATUS FOR PRODUCING VIBRATIONS.

Fig. 1 is a diagrammatic sketch of the principal parts of this apparatus. Two rods of wood, a and a' , are free to turn about b and b' respectively. At c and c' these rods are connected by means of bolts, free to move in slots, to a third rod d which slides in a guide on each side. The rod d is connected as shown to a 12 in. pulley P . By changing the point of attachment the amplitude of vibration could be changed. The pulley

¹ N. Cimento, 14, pp. 537-564, 1907.

² PHYS. REV., 28, Feb., 1909.

³ Phil. Mag., 13, pp. 635-643, 1907.

⁴ Amer. Acad. Proc., 11, pp. 321-343, 1911.

P was driven by a small motor and the number of revolutions of this pulley was determined by means of a revolution counter attached to the axle. This gave a convenient method for determining the number of vibrations as each revolution of P produced a vibration in the wire. The two small pulleys p and p' were connected to the rods a and a' by means of strong cords wrapped two times around the pulleys. The axle of pulley p turned in a block firmly fastened to one end of the table to

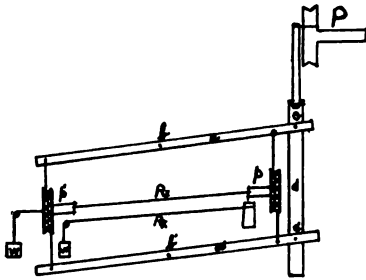


Fig. 1.

which the entire apparatus was fastened and pulley p' was similarly fastened to the other end of the table. The wire to be vibrated was securely clamped into the axes of these small pulleys by means of set screws threaded into brass rods which formed a part of the axes. A cord from the center of p' ran over a pulley and carried a weight w . This produced a known tension in the wire as p' could always be so placed as to

make w the tension on the wire, while p had a collar pressing against the block in which it turned. R_v is the wire which was vibrated and R_k was used as the constant reference resistance. R_k was mounted parallel to R_v , the two being about one centimeter apart.

RESISTANCE MEASUREMENTS.

In this work it was not necessary to know the actual resistance, the per cent. change resulting from the vibrations being the information sought. A potentiometer method was used for this purpose as it removed errors due to contact resistance. This is an essential consideration as even small variations in contact resistance would conceal the true changes where new contacts have to be made for each reading. In order to make changes in the temperature less effective and in fact to keep the reference resistance and the one vibrated the same in all respects, except for the vibrations, a wire of the same material and dimensions and having received the same treatment before the experiment as the vibrated wire, was used for the constant resistance. The electrical connections are shown in Fig. 2 in which R_v is the vibrated wire and R_k the one kept constant. These were mounted as described above and when readings were taken they were connected in series with three storage cells in parallel by means of the reversing switch S_1 . The potentiometer used was a Pye potentiometer reading directly one part in ten thousand. This potentiometer and the resistance R and r were connected in series

with a storage cell by means of the reversing switch S_2 . Two points on R_k were brought to one side of a double pole, double throw switch S_3 and two points on R_x were brought to the other side of the same switch. The middle points of S_3 could be put in series with the galvanometer G by means of a similar switch S_4 . The points could thus be brought to the potentiometer through the galvanometer. Since R_x and R_k were in series the ratio of the readings on the potentiometer for R_x and R_k gave the ratio of these resistances.

Some of the work was done at temperatures as high as 450° C. For this purpose an electric furnace was made through which the wires R_x and R_k could be passed. The high temperatures were determined by means of a copper-advance thermocouple which had been calibrated

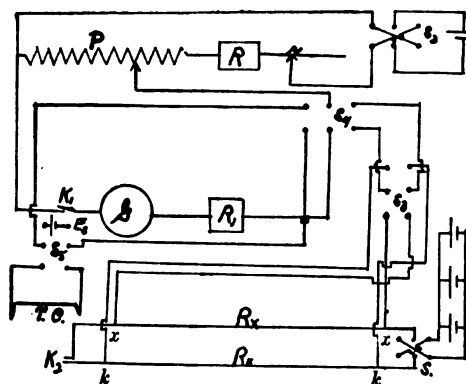


Fig. 2.

in terms of a cadmium standard cell and the potentiometer above mentioned. By means of the switches S_4 and S_6 the standard cell and the thermocouple could be brought to the potentiometer when desired.

In order to keep the wires R_x and R_k free from unknown strains, no permanent connections were clamped or soldered on R_x or R_k at the points that were brought to the potentiometer. Moreover such connections would have been quite inaccessible when the wires were in the furnace. It is evident that the accuracy of the work cannot be greater than the accuracy with which the points kk and XX can be duplicated. With this fact in mind the following method for determining the points kk and XX was devised.

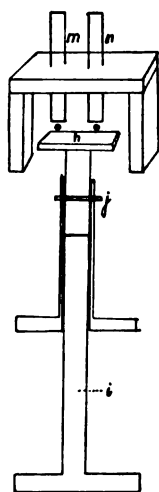


Fig. 3.

A frame made of heavy wood was firmly screwed to the table on which the vibrating apparatus was placed. Two $\frac{3}{8}$ in. brass rods, m and n , were run through the upper piece of the frame and were held in a fixed position by passing through iron plates which were screwed to the top and bottom of the wood and were provided with set screws. The rods m and n projected over the wires R_x and R_k respectively (see Fig. 3), but did not touch the wires except when the insulating plate h was raised. The plate h ,

made of insulating material that would stand high temperatures, was screwed to the top of a $\frac{3}{8}$ in. brass rod which fitted into a metal tube and was kept from turning by means of the metal pin, j , which moved in a slot in the tube. The tube was fastened to the table top and the rod which carried the plate h was supported on a rod i , which threaded into the bottom of the tube and projected through a hole in the table. Thus the wires could be pressed against the rods m and n and released again by simply turning the rod i . A similar arrangement was used to make connections with the rods m' and n' . The rods mm' and nn' projected into the furnace, as did the rods which carried the plates h and h' .

METHOD OF TAKING READINGS.

Straight wires were mounted and put under the desired tension. The tension in R_s and R_k was always the same. It was necessary to take proper precautions not to mount the wires under a torsional strain, especially when soft annealed wires were used. The ratio of R_s to R_k was determined as described above. In determining this ratio at least four readings were taken for each wire. This was found advisable as the currents seldom stayed constant within the range of sensibility. By taking a reading, say for R_s then for R_k , as quickly as possible the change in the currents was small and by then taking readings in the reverse order the effect of change of current was made a minimum. The direction of the current was then reversed and similar readings taken; this tended to minimize the effect of local thermal E.M.F.'s. The average of all these readings was taken to determine the ratio of R_s to R_k . No current flowed through R_s and R_k while R_s was vibrated and during this process the wires were free from the rods and plates used to make connections. After a known number of vibrations the motor was stopped and the pulleys were set as at the start and the ratio of R_s to R_k was again determined. The per cent. change in this ratio gave the per cent. change in R_s due to the vibrations, as R_s and R_k were treated alike in all other respects.

SENSIBILITY AND ACCURACY.

It was found that the readings could not be duplicated to a higher degree of accuracy than one part in eight thousand. The sensibility could be made much greater but this was not necessary as long as the accuracy was no higher, hence in most of the work conditions were so chosen as to give the readings on the potentiometer a value of about eight thousand. Although the method used to make the points XX and kk definite was far the best of several tried it still leaves much to be desired, as changes less than .02 per cent. could not be determined. If

the cords used for belts on the pulleys p and p' slipped so as to give R_z a permanent torsional strain the ratio of R_z to R_s always increased, but this slipping was easily overcome by having the cord make two complete turns around the pulleys and by a liberal use of rosin on the cords and pulleys. That this was accomplished is evident from the fact that in more than a million vibrations there was no noticeable change in the relative positions of marked points on the pulleys. The original position of the pulleys could easily be duplicated by making it come at the end of the stroke of the driving rod, for in this position the large pulley could be turned several degrees and still not change the positions of the small pulleys.

The errors due to permanent changes in the dimensions of the wires could be detected by observing the displacement of p' . The distance of this pulley from the fixed block in which it turned was measured by means of calipers and a change of .2 mm. could be detected. Such a change would not produce a change in resistance sufficient to be detected under the conditions but whenever there was any evidence of a displacement of p' the readings were discarded.

RESULTS.

The length of the wire R_z between the points of attachment in the pulleys p and p' was 122.3 cm. and the points on the wires brought to the potentiometer were 63.3 cm. apart. The amplitude of vibration, referred

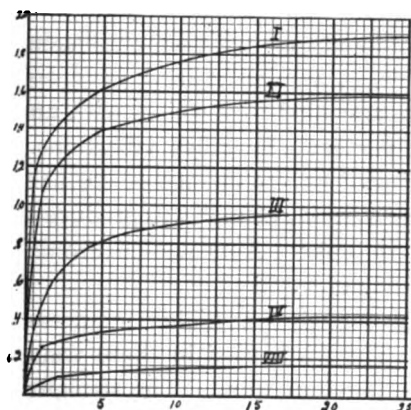


Fig. 4a.

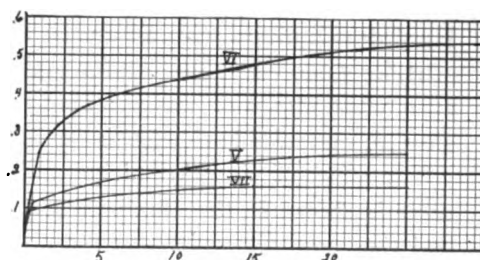


Fig. 4b.

to below, will always mean the angle through which each end of the wire was turned. Wires were annealed under zero tension, by passing an electric current through them sufficient to raise them to bright red heat. The wires used were taken from the research stockroom. Most of them

had been purchased from Eimer and Amend or The Driver-Harris Wire Company. Since nickel showed the largest changes, wires marked "pure" from the two firms were tested and no difference in the results was observed.

The curves in Fig. 4*a* and Fig. 4*b* show the per cent. increase in resistance produced in the different materials under the following conditions:

Curve.	Material.	Diameter of Wire, mm.	Tension, lbs.	Amplitude of Vib.
I.	Annealed Ni.....	1.01	5	335°
II.	Soft drawn Ni.....	1.00	5	335
III.	Annealed Cu.....	0.88	1.5	335
IV.	Unannealed Cu.....	0.88	3.0	335
V.	Annealed manganin....	1.00	6.25	598
VI.	Unannealed nichrome...	.80	6.00	335
VII.	Annealed platinum.....	0.62	2.00	335
VIII.	Annealed brass.....	1.18	6.25	335

The temperature was 22° C. in all the work where no other temperature is given. The ordinates for all the curves are the per cent. increase in resistance and the abscissas are thousands of vibrations.

Piano wire one mm. in diameter, under tension of 10 lbs., annealed and unannealed, gave no measurable change with an amplitude of 335°. Annealed wire with an amplitude of 598°, gave for one thousand vibrations an increase of .10 per cent. and this rose to .22 per cent. for 12,000 vibrations but remained constant beyond this. Annealed and unannealed manganin wire gave no measurable change with an amplitude of 335°. Phosphor-bronze (1 mm. diameter, tension 5 lbs.) gave no measurable change in any test except the annealed wire which, with an amplitude of 598°, gave a change of .04 per cent. for 4,000 vibrations. Hard drawn nickel with an amplitude of 500° gave a change of .29 per cent. for 5,000 vibrations but for 335° no measurable change was produced even when the wire was vibrated until it broke. Annealed nichrome gave no measurable change for an amplitude of 335°.

The effect of vibrating nickel and nichrome wires at higher temperatures is shown in the curves of Fig. 5. The wires used in these tests had the same dimensions as those in the previous work. The straight vertical lines drawn at the ends of the curves for nickel show the per cent. change in the resistance when the temperature was allowed to fall to room temperature. In every case the change represented by the vertical line disappeared when the original higher temperature was restored. Nichrome did not show such changes and it is evident from these curves that high temperatures produce quite opposite effects in nichrome and in nickel.

For temperatures above 350°C . annealed nickel becomes soft so that the wires were permanently lengthened by the tension used and the change in dimensions accounts for the change in resistance. Annealed copper wires had to be handled carefully at room temperature to prevent permanent changes in their dimensions and when the temperature was raised to 100°C . the results were not free from this error. A certain tension was necessary to hold the wires in position and this minimum tension at the higher temperatures made the results unreliable. Manganin and piano wires were vibrated at temperatures as high as 350° but no measurable change was produced.

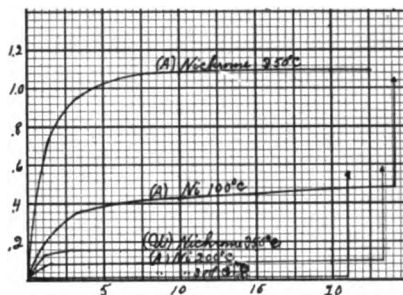


Fig. 5.

In all cases the change in resistance, due to vibrations, could be removed by annealing the wires at bright red heat. Raising them to 500°C . would remove most of the change while at this temperature, but the resistance would rise to an intermediate value when the temperature was lowered. Fig. 6 curve I. is the result of vibrating an annealed

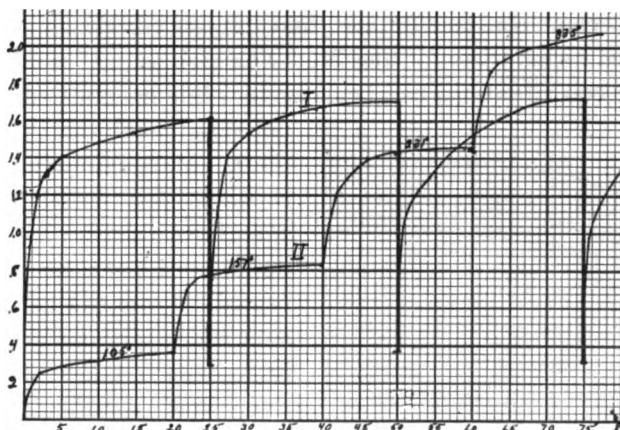


Fig. 6.

nickel wire at room temperature until the resistance had become practically constant. The wires were then raised to bright red heat under zero tension and when cooled the resistance dropped to near its original value. This was repeated as shown and the fact that the original value was never quite reached after annealing was undoubtedly due to the fact that the temperature was not raised high enough.

The effect of amplitude of vibration has been indicated in the case of manganin and piano wires. Curve II. in Fig. 6 shows this effect for annealed nickel. These results were obtained by vibrating the same wire, simply changing the amplitude as indicated. In Fig. 7 this effect is shown for different nickel wires which had received the same treatment except for amplitude of vibration. The largest amplitude approaches the yield point of the wire and in this case the resistance changes until

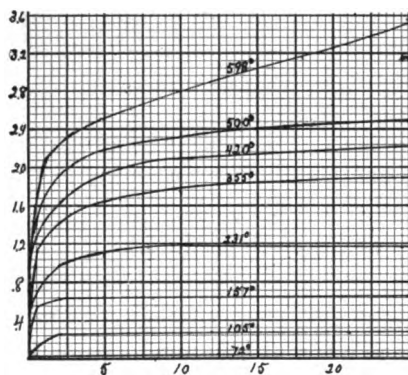


Fig. 7.

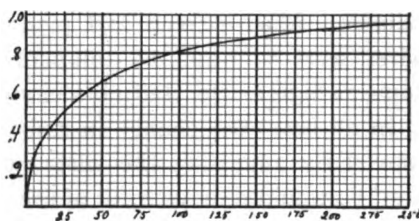


Fig. 8.

the wire breaks. For amplitudes of 335° and above the resistance does not become constant for less than 100,000 vibrations.

Annealed nickel wires for amplitudes as high as 335° would not break before 200,000 vibrations had been made, while unannealed wires would

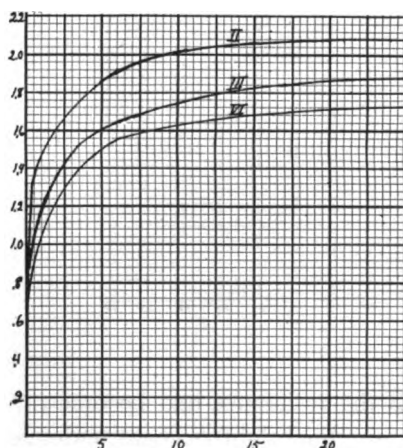


Fig. 8a.

not stand more than 50,000 vibrations. Manganin wires have about the same life as nickel, but nichrome wires have a shorter life, as they are quite hard even after being annealed. From the curves it is evident that a large part of the change in resistance takes place in relatively few vibrations. The curve in Fig. 8 gives the shape of the curve for the first 250 vibrations for annealed nickel.

The effect of tension for annealed nickel is shown in Fig. 8a, curves II., III. and IV., the tension being 8, 5 and 2.6 lbs. respectively. Similar effects were observed for other materials.

In all the above work, for amplitudes below 500° the rate of vibration

was approximately 150 per minute and 100 per minute for the other amplitudes. The change in rate of vibration for the range of 60 to 180 per minute was found to have no measurable effect on the results. The motor used would not drive the apparatus at a lower speed than 60 vibrations per minute and 180 vibrations was the highest speed that could be used except for very small amplitudes.

In their work with platinum-iridium wire Guthe and Sieg,¹ and later Sieg,² found this alloy to possess peculiar elastic properties. These properties are especially marked when the percentage of iridium is high. Through the kindness of Mr. Sieg a 40 per cent. platinum-iridium wire was made available for the present work. Since this wire, the available length of which was 33 cm., was much shorter than those used above, the apparatus had to be changed and the accuracy was decreased. It was also less certain that the observed change was entirely due to the vibrations as the reference wire had to be of some other alloy. Another source of difficulty was the small diameter (.2 mm.) of the wire, and since the wire was to be used in other work care was taken not to injure it. For these reasons the following results cannot be taken as a fair comparison with those on other materials. The annealed wire under a tension of one pound for an amplitude of 105° gave an increase of .06 per cent. for 1,000 vibrations. It was again annealed and the amplitude was changed to 335°, which gave the results indicated below.

Total Vib.	Per Cent. Increase in R.
100	.31
500	.63
2,000	.75
3,000	.81

It is hoped that later this investigation may be extended for the purpose of comparing the changes in electrical resistance with the changes in elastic properties. In the present work there was no evidence of any tendency for the wires to return to their previous resistance after being vibrated and then allowed to rest. In many of the tests wires were vibrated a short time then allowed to rest for a day or two but no change could be detected due to the resting. It is also hoped to try the effect of lower temperatures.

The writer wishes to take this opportunity of acknowledging his obligations to Professors Nichols and Shearer for their continued interest and helpful suggestions throughout this work, which was undertaken at the suggestion of the latter.

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CORNELL UNIVERSITY.

¹ Guthe and Sieg, *PHYS. REV.*, 30, p. 610, 1910.

² Sieg, *PHYS. REV.*, 31, p. 421, 1910.

REFLECTION OF β RAYS BY THIN METAL PLATES.

BY WILLIAM B. HUFF.

IT is known that α particles may pass through a thin metal foil, for example, and emerge with decreased speed and scattered by the atoms of the metal. The researches of Geiger¹ have led to remarkably definite results on these two points.

Analogous problems as to scattering and loss of speed are also offered in the case of β particles incident upon thin plates. Here also it is the atom that does the scattering, as is shown by the work of Crowther² and of Schmidt.³ In addition to β particles that pass through the plate and emerge scattered by the atoms, there are some that, scattered and losing all speed, are absorbed. Schmidt³ and Borodowsky⁴ have shown that this absorption depends upon the atoms alone and is independent of chemical combination and of physical state of the absorbing matter.

In the determination of loss of speed shown by β particles that have passed through a thin metal plate, the experimental difficulties are much greater than in the corresponding problem for α particles, where even a single particle and its effect can be studied. Magnetic sorting secures only approximately homogeneous pencils of β rays; they are easily scattered; their average range is long; and their ionization per centimeter of path is extremely low.

Nevertheless Wilson⁵ has given reasons for believing that β particles show a loss of speed as a result of passing through a metal plate, and the photographs by von Baeyer, Hahn, and Meitner⁶ appear to confirm his results.

The scattering effect produced by the atoms is shown not only by the particles that pass through the metal plate; some may be deflected through such large angles as to emerge on the side of incidence.

The importance of the atom in this reflection of β particles was recognized early, but the introduction of the idea of a true secondary β radiation, having its origin in the reflector, tended to render difficult the explanation of observations.

¹ Roy. Soc. Proc., 83, 1910.

² Roy. Soc. Proc., 84, 1910.

³ Phys. Zeit., X., 1909; XI., 1910.

⁴ Phil. Mag., 19, 1910.

⁵ Proc. Roy. Soc., 84, 1910.

⁶ Phys. Zeit., XII., 1911.

Earlier results, as well as the recent ones of Gehrts,¹ show that electrons striking a metal plate may be reflected, and may also give rise to a true secondary. In the case of β particles reflected under ordinary conditions, Madsen² showed that scattering could account for observed effects.

REFLECTION CURVES.

Thus for both kinds of incident particles, the reflection is due to the atoms. For a given source, the amount of reflected radiation depends also upon thickness of reflector; and more markedly in the case of β particles because of their greater penetrating power.

Curves showing increase of ionization as the metal plate reflecting β particles into an ionization chamber is increased in thickness may be called reflection curves. Among those who have given such curves are Schmidt,¹ Madsen,² and Pound.³ The present writer⁴ has given some typical curves, with the addition of absorption curves to show difference in quality of the reflected radiation.

Since reflection is due to the atoms, special interest attaches to differences between reflection curves for substances of widely different atomic weights. It is well known that variation of experimental arrangement may produce considerable modification in the forms of these curves. *A* and *B*, Fig. 1, may indicate the general features of the reflection curves for lead and aluminium respectively.

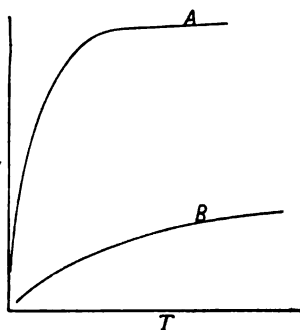


Fig. 1.

T represents thickness of reflector; *I*, the ionization due to reflected particles.

A comparison of the curves shows that, for a given thickness of metal, the radiation returned by the lead produces the larger ionization. Further, the thickness of reflector that returns practically all the ionizing radiation obtainable from each reflector is not only smaller for lead than for aluminium, but also more definitely indicated on the curve. A comparison of the quality of the maximum radiation reflected from each of the two metals shows that from lead to be the more penetrating.

¹ Ann. d. Phys., 36, 1911.

² Phil. Mag., 18, 1909.

³ Ann. d. Phys., 23, 1907.

⁴ L. c.

⁵ Phil. Mag., 17, 1909.

⁶ Phys. Rev., 30, 1910.

THIN REFLECTORS.

If β particles that pass through a metal plate emerge with decreased speed, then some of the particles reflected by a thick plate may be expected to show a similar loss. Accordingly, the β radiation returned by a thin reflector should be more penetrating than that from a thicker reflector of the same material. Such a difference in penetrating power would be further evidence that these particles lose speed in passing through matter.

As stated in the account of the writer's experiments referred to above, consistent results for reflectors as thin as a few gold-leaves were not obtained and the initial parts of the reflection curves were left undetermined.

The present paper records some observations on the amount and quality of the β radiation reflected by the thinnest obtainable metal plates.

APPARATUS.

The method was a simple modification of one that is well known. A diagram of the apparatus is shown in Fig. 2.

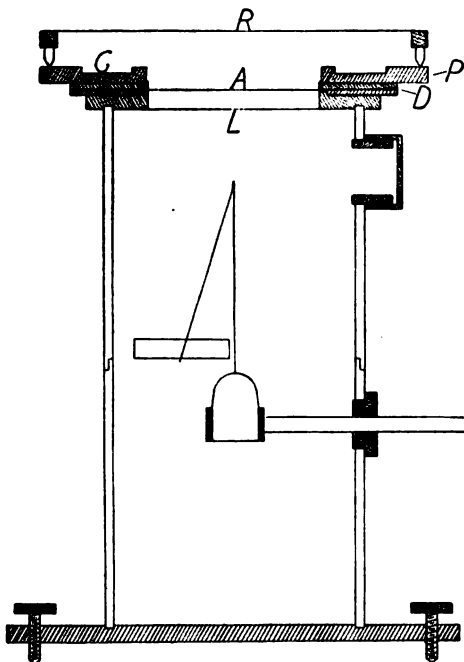


Fig. 2.

The cylindrical brass-walled electroscopical case contained dry air, and the gold-leaf was insulated with sulphur. A circular opening 4 cm. in diameter in the top of the case of the electroscopical case was closed with a single aluminium leaf *L*. Layers of absorbing material *A* could be placed between two thin brass rings *D*, close to *L*. In the brass plate *P* was a shallow circular groove *G* to contain the radioactive material, which was thus distributed around the vertical axis of the electroscopical case. The circular reflector *R* was 8 cm. in diameter and was supported 1 cm. above the plane of *P* by a slender brass tripod. It is probably unnecessary to refer to the extreme im-

portance of reducing as far as possible the effect of temperature changes within the case of a delicate electroscopical case used for comparing weak

ionizations. A felt-lined box with proper windows and enclosing the entire apparatus was found to answer the purpose.

It was desired to obtain measurements of ionizations due to particles reflected by the thinnest metal sheets; and from readings in which a considerable part of the observed ionization should be due to the reflected particles. These considerations determined the position of G as well as the area of R and its distance from G . The effect of very penetrating radiation directly from G to the electroscope was small. Corrections were made by taking readings with R removed, but with everything else in position, including absorbers. Multiple reflections, both before and after particles entered the electroscope, undoubtedly modified the results. Wherever possible, metal was covered with cardboard.

Since particles from R reach the absorber at A with very different angles of incidence, numerical results on coefficients of absorption would mean little. Further, sets of tubes designed to render parallel the rays entering an ionization chamber may introduce, especially in absorption experiments, large effects due to reflection and scattering.

Without going into detail, it may be stated that a set of parallel aluminium tubes was placed between a thin layer of uranium oxide and an electroscope and the ionization measured. Replacing the aluminium tubes by an exactly similar set but made of lead, the ionization was increased 45 per cent. Again, using brass tubes, and a thin sheet of tin as absorber, the ionization could be doubled by changing the position of the tin only.

Ur-X was chosen as the source, since Schmidt's¹ work seemed to justify the assumption that for this substance the initial velocities of the β particles were, within narrow limits, the same.

These limits were probably widened in the present experiments by reflection from the bottom of G and by the 0.1 mm. of aluminium that covered the Ur-X.

CURVES.

For each set of curves shown in the following figures, the ionization is in arbitrary units; the thickness of reflector or of absorber, in millimeters.

Fig. 3, a , was got by using tin to absorb β particles reflected from a tin plate 3 mm. thick. These same absorbing layers were used for b and c . For b , the reflector was a sheet of tin 0.022 mm. thick; for c , three of these sheets.

Comparing ionizations represented by corresponding points of a and b , it is seen that initially the effect of the thin reflector is one third that of the thick one. This ratio of ionizations falls as the absorbing layer

¹ Phys. Zeit., 10, 1909.

is made thicker, and then rises for still thicker absorber. A comparison of tangents at corresponding points shows that the radiation reflected from the thick plate is the softer.

Fig. 3, *d* and *e*, show absorption by lead of radiation returned by the thinnest and thickest tin reflectors.

A set of curves analogous to those of Fig. 3 was obtained for lead as reflector and absorber. With no absorber except the aluminium leaf that closed the electroscope, the rays reflected from a sheet of lead 0.033 mm. thick gave an ionization 63 per cent. as great as that got by using a

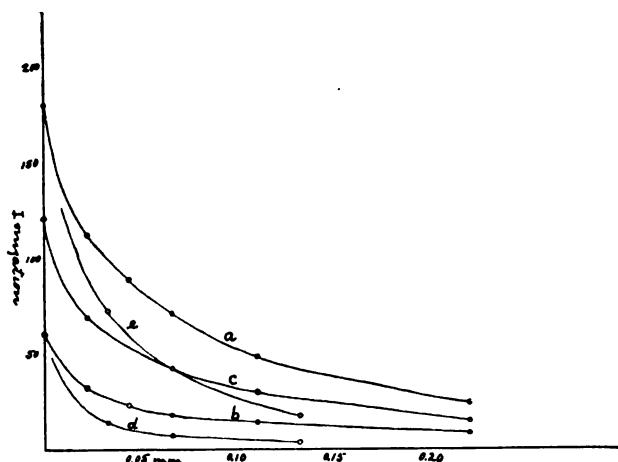


Fig. 3.

3 mm. reflector. As in the case of tin, these absorption curves for lead showed that radiation from the thick reflector was, as a whole, more easily absorbed than that from the thin sheet.

Thus as tested by the absorption produced by metal foil close to the small ionization chamber, the effect of decreasing the thickness of the reflector is not only to decrease the total amount of reflected radiation, but also to change the quality; the average penetrating power of particles from the thin reflector is greater than that of particles returned by the thick plate.

With reference to the change produced in the form of the absorption curve, the result of making the reflector thinner may be described briefly as a less rapid initial drop and a quicker approach to linearity.

Gold foil was used as still thinner reflector and typical of metals of high atomic weight. Average thickness was determined by weighing and assuming density.

The initial parts of a reflection curve got by using this foil are shown

by Fig. 4, *a*, where the maximum thickness of gold used gave about 48 per cent. of the maximum ionization obtainable from a thick reflector of this metal, the leaf closing the electroscope being the only absorber in each case.

The rapid initial rise is followed by a rate of increase nearly linear. Succeeding this linear part of Fig. 4, *a*, the complete reflection curve, as indicated in Fig. 1, *A*, would show the ionization approaching the constant value corresponding to the maximum effect of the radiation returned by the thick plate.

Fig. 4, *b* and *c*, show the absorption by this gold foil of particles reflected from a single layer of the foil and from ten such layers respectively. A comparison of these two curves brings out the approach to linear absorption as the reflector is made thinner.

For *e* and *d* of Fig. 4 the reflectors were respectively one and ten layers of the gold foil, and the absorber was aluminium. After the slight initial drop, the results are linear within the limits of accuracy of measurement. Thus β particles reflected by the single foil are slightly absorbed by aluminium. The difference in slope of *d* and *e* shows that the additional sheets of foil increased not only the more penetrating radiation, but, to a larger extent, that which was more easily absorbed.

This more rapid rate of increase of the softer radiation as the reflector is made thicker may be correlated with Kovarik's¹ observations on coefficient of absorption measured first for incident β particles alone, and then for incident and reflected particles together; the combined radiation was softer than the incident alone.

The resemblance between such a curve as Fig. 4, *a*, and the one for scattering of α particles has been pointed out by Geiger.²

Finally, leaves of gold, copper, and aluminium were used as reflectors. The determination of thickness of such leaves by weighing is an uncertain process. Even when the leaves are chosen so that they have nearly equal weights, any leaf shows inequalities in thickness, and these were especially marked in the case of the copper, for which no consistent observations were obtained.

When the first leaf was fastened firmly to the upper surface of the ring used to support the reflectors, the additional leaves could be piled closely and held in position with soft wax.

There is also uncertainty in the measurement of ionizations so small that variations of the natural leak become important.

The initial part of the reflection curve of aluminium is shown by Fig.

¹ Phil. Mag., 20, 1910.

² Proc. Roy. Soc., 82, 1909.

4, *f*. Determination of the position of the first few points is subject to such large probable error that it may not be said that they do more than suggest inflection. The curve then becomes closely linear.

The reflection curve for a thickness of gold up to thirty-one leaves is shown in Fig. 5, *a*, this maximum thickness being roughly equivalent to two layers of the foil used for Fig. 4, *a*.

The increase of ionization is initially nearly linear. The effect due to adding the second reflecting leaf arises from particles scattered and

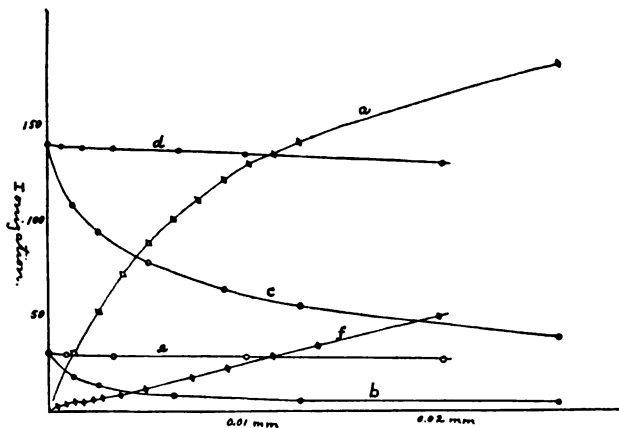


Fig. 4.

retarded by the first leaf. The scattering would decrease the number of particles that enter the electroscope; the speed-loss would render more efficient ionizers those that do enter.

The absorption by gold leaves of particles reflected by eight and by thirty-one gold leaves is shown by *b* and *c*, Fig. 5. Corresponding parts of the two curves show different rates of absorption; in each curve the initial rate of absorption is greater than the final one; and in neither case is there a sudden initial drop.

Points for the absorption of particles reflected from a single gold-leaf were too uncertain to do more than indicate the linearity to be expected from *c* and *b*.

As might be inferred from *d* and *e* of Fig. 4, aluminium absorption of particles reflected from a few gold-leaves was linear.

It is assumed that such absorptions as those indicated are of reflected β particles, and that the effects of γ rays traversing the reflectors are negligible.

As already pointed out, the initial irregularities of the reflection curves, Fig. 4, *f*, and Fig. 5, *a*, are probably due to errors of measurement, to-

gether with the fact that the air above the leaves is also a reflector. The question of homogeneity of the incident rays becomes especially important for such reflectors as a few leaves of aluminium, which would return an excess of the slower and therefore more strongly ionizing particles.

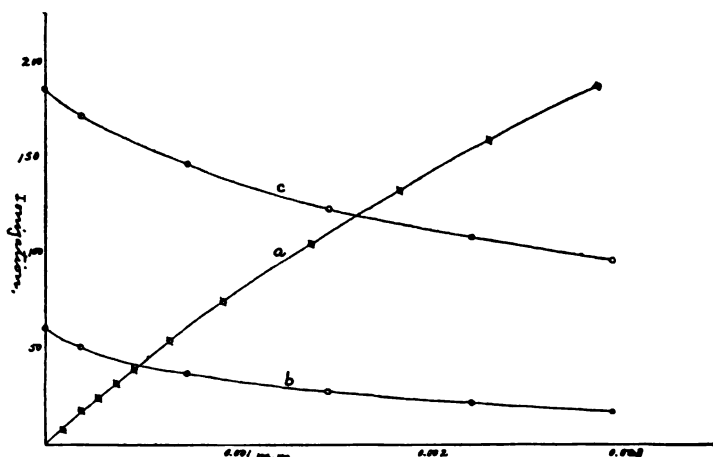


Fig. 5.

Experiments are being made with a source the β rays of which possess groups of velocities, as shown in the photographs by V. Baeyer and Hahn.¹

THICK REFLECTORS.

Because of its heavy atoms, a thin sheet of gold reflects not only more of the incident particles than an equal thickness of aluminium does, but also a larger proportion of the swifter particles; since the incident β rays are not strictly homogeneous. Consequently, there is a difference between the quality of the radiation transmitted in the two cases. Further, the scattering of transmitted particles is more nearly complete in the case of the heavy atoms, and any loss of speed due to increase of path operates to make still more easily reflectible the particles that have passed through the thin gold sheet. Accordingly, because of difference in quality of the rays transmitted to additional reflecting sheets, as well as because of difference in reflecting power of layers of light and heavy atoms the radiation transmitted once in the direction of incidence would vanish for fewer layers of gold than of aluminium.

Multiple reflections also occur within the reflector. These increase the thickness of metal necessary to give the maximum emergent radiation, but still leave this thickness less, and more sharply defined, in the case of lead or gold than for aluminium.

¹ Phys. Zeit., XI., 1910.

It is found that a single gold-leaf reflects a measurable amount of the incident radiation. Now add a second reflecting leaf and the particles reflected by it are scattered and retarded by the first one before they can enter the electroscope. This scattering reduces the number of particles that enter and the retardation increases their ionizing power. There are also multiple reflections between the two leaves and these would result in some absorption, especially of slower particles between two layers of heavy atoms.

Additional layers give further points on the reflection curve, and where the ionization increases linearly, loss of ionizing particles due to scattering and absorption is compensated for by the lower speed of those that reach the ionization chamber. The differences in the initial parts of two reflection curves such as those indicated by Fig. 4, *a* and *g*, may thus be accounted for.

Thus the large range of speeds observed in β particles reflected by a thick plate upon which nearly homogeneous rays are incident may be regarded as due chiefly to loss of speed within the reflector.

Because of difference in reflecting power of light and heavy atoms, those particles that emerge from a thick aluminium reflector have had an average longer path in the reflector than similar particles could have in gold and still emerge as reflected particles.

In a general way, the differences between *A* and *B*, Fig. 1, may thus be accounted for, as well as the difference in the quality of the radiations reflected by metals of widely different atomic weights.

SUMMARY.

I. The thinnest obtainable metal plate may reflect a measurable amount of the incident β radiation. As measured by its ionization in a small chamber, the increase of reflected β radiation is initially closely proportional to the thickness of reflector.

II. The quality of the reflected β radiation is dependent on thickness of reflector. The particles reflected from a thin reflector are absorbed linearly by thin sheets of a metal of low atomic weight. The absorption appears to approach linearity when both reflector and absorber are of high atomic weight but extremely thin.

III. The difference in quality shown by β radiation from thin and from thick reflectors of the same metal may be regarded as evidence of loss of speed due to passing through matter.

A SPONTANEOUS ELECTROMOTIVE-FORCE IN CELLS OF ALKALI METALS.

BY JAY W. WOODROW.

IN working with photo-electric cells, it was found that when the cell was insulated in the dark the alkali metal would develop a negative charge. As all attempts to remove this effect were unsuccessful, it was decided to investigate the phenomena more carefully. The results of these experiments will be described in the following paper.

The apparatus was set up as shown in Fig. 1. The alkali metal was placed in the cell *C* and as high a vacuum as possible obtained by means of a Gaede pump so that a perfectly pure surface could be procured. The different metals investigated were cæsium, potassium, and an alloy of sodium and potassium formed by melting together masses of the two proportional to their atomic weights respectively. Throughout the investigation, the cell was kept carefully screened from the light by the box *D*.

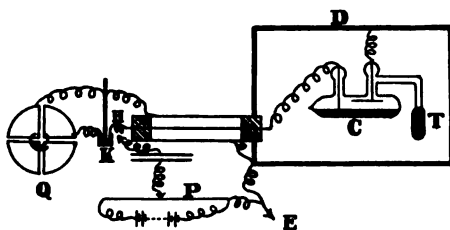


Fig. 1.

In measuring the potential, the electrode of the alkali metal which was connected to one pair of quadrants of the electrometer *Q* was insulated by raising the plunger of the key *K*. The other electrode and the other pair of quadrants were earthed. The maximum potential was determined from the deflection of the electrometer which had a sensitiveness of 160 mm. per volt.

In order to measure the current which the cell would give, the interior of a small cylindrical condenser (capacity = 24.55 E.S.U.) was connected in as shown at *H*. The outer cylinder of the condenser was connected to a potentiometer by means of which any desired potential could be obtained. After raising the plunger of the key *K*, the potential on the exterior of the condenser was gradually increased so as to keep the electrometer always at the zero position. From the known potential and capacity, the charge drawn into the condenser in a given time was determined and from this the current was readily calculated.

MEASUREMENT OF THE POTENTIAL.

The plunger of the key *K* was raised and the electrometer readings were taken at regular intervals. At first the deflection increased rapidly, but after a few hours the reading became constant. The curve obtained by plotting potential against time resembles a saturation curve. Typical cases are shown in Fig. 2. Curve I. was obtained with a cell in which the electrodes consisted of an aluminium wire placed parallel to the surface of the alloy of sodium and potassium. The cell from which Curve II. was obtained contained caesium and platinum electrodes. The time required for the cells to attain the maximum potential was found to depend upon the size and distance between the electrodes. In one cell in which the earthed electrode was simply the end of a platinum wire, it required over thirty hours to attain the maximum potential; while the

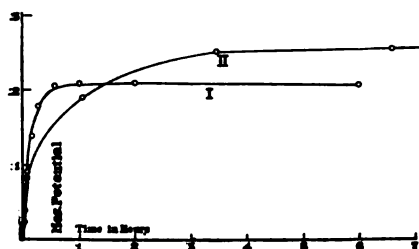


Fig. 2.

other cells in which larger electrodes were used reached the maximum value in from two to four hours. If instead of earthing the aluminium (or platinum) electrode, it was kept at a constant positive potential equal in value to the negative potential attained by the alkali metal, the deflection was reduced to zero.

Five different cells with the sodium-potassium alloy for the alkali electrode were investigated and the mean maximum negative potentials in volts are given in Table I. Cell III. was constructed with two elec-

TABLE I.

Number of Cell.	Maximum Potential Attained.			Remarks.
	Charcoal at 25° C.	Charcoal in Liq. Air.	No Charcoal Tube.	
I.	2.2	1.5		Al electrode earthed.
II.	2.0			Al electrode earthed.
III.			1.5	Al electrode earthed.
			1.5	Pt electrode earthed.
IV.	2.4			Al electrode earthed.
	2.5			Tube filled with hydrogen.
V.	2.8			After electric discharge.
			1.6	Al electrode earthed.
Caesium			2.4	After electric discharge.
Potassium	2.6			
	2.0			

trodes, in addition to the platinum electrode which was in contact with the alloy, so that measurements could be taken with either an aluminium

or platinum electrode earthed while at the same time no change would be made in the connections between the alloy and the electrometer. It will be seen that the maximum potential obtained in the two cases was identical. If the effect had been due to a contact potential difference between the alloy and the platinum or aluminium,¹ the magnitude of the potential in the two cases would have differed as the contact difference of potential between potassium and aluminium is 1.9 volts while that between potassium and platinum is 4.1 volts.² Cells exactly similar to the above in construction, containing clean mercury in a high vacuum, showed no leak which could be measured when treated as were the cells of alkali metals. It is also worthy of notice that the mean maximum potential obtained with Cell I. when the charcoal tube was immersed in liquid air is exactly equal to that obtained with Cell III. to which no charcoal tube was attached.

Interesting phenomena were shown by Cell IV. This cell contained strips of palladium which had been made to absorb great quantities of pure hydrogen by using it as the cathode in an electrolytic cell of dilute sulphuric acid. Upon heating the palladium the tube was filled with hydrogen, but very little change was observed in the maximum potential. However, upon passing an electric discharge from an induction coil through the cell for several minutes, the potential was increased quite perceptibly. This increase of the potential would disappear in a few days but could be renewed by means of another discharge. This effect of the electric discharge was even more pronounced in another cell, No. V., which contained no hydrogen. It was rather surprising to find that although the hydrogen was shown to exert considerable pressure within the tube, yet it seemed to make no change or at least very little change in this effect.

EFFECT OF TEMPERATURE UPON THE MAXIMUM POTENTIAL.

The effect of the temperature upon this potential was next investigated. The cell was heated by means of an electric furnace in which the wires were so wound as not to produce a magnetic field. The temperatures were read on a good mercury-in-glass thermometer suspended so that the bulb was in contact with the glass walls of the cell. This of course gave only a close approximation to the actual temperature of the alkali metal but was thought to be sufficiently accurate for the purpose of the experiment. The temperatures below room temperature were obtained by partially immersing the cell in ice and a mixture of salt and ice which was

¹ Cf. Jakob Kunz, *Phys. Rev.*, XXXI., p. 538, 1910. Cf. F. K. Richtmyer, *Phys. Rev.*, XXIX., p. 74, 1909.

² Winkelmann, *Handbuch der Physik*, Vol. 4, p. 855.

contained in an earthed tin vessel. A set of values is given in Table II. for the mean potentials observed with one cell. The potentials are recorded in volts.

TABLE II.
Potential as a Function of the Temperature.

Temperature.	Maximum Potential.	Temperature.	Maximum Potential.
-15° C.	2.00	60° C.	2.18
0° C.	1.92	65° C.	2.24
15° C.	1.20	75° C.	2.21
25° C.	0.73	80° C.	2.20
30° C.	0.91	105° C.	2.13
40° C.	1.59	120° C.	2.02
50° C.	1.91	160° C.	2.00

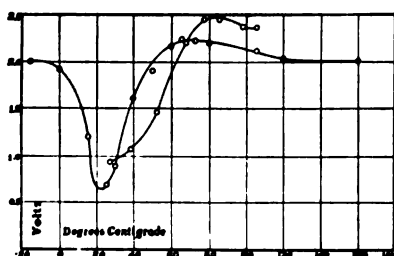


Fig. 3.

The curves in Fig. 3 both indicate that the minimum effect is obtained at about 25° C. and the maximum at about 75° C.; while above the latter the potential decreases slightly and then remains very nearly constant with further increase in the temperature. At the higher temperatures the maximum potential was reached

in a few minutes after insulation. This fact led to an investigation of the current which the cell would give.

EFFECT OF TEMPERATURE UPON THE CURRENT.

The current was measured as explained earlier in the paper, and the desired temperatures were obtained as in the previous case where the potential was determined. In Fig. 4 are shown curves which represent the current given by two different sodium-potassium cells. These curves are seen to be identical with those which Richardson¹ obtained for the current between a hot platinum wire and a metal cylinder surrounding it in a high vacuum. That is, for temperatures above 30° C., the current given by a cell with electrodes of platinum and an alloy of sodium and potassium in a high vacuum will when carefully screened from light be represented by

$$i = aT^{\frac{1}{2}}e^{-b/T},$$

where a and b are constants and T is the absolute temperature. The following constants were calculated for Cell No. VI.,

¹ O. W. Richardson. Proc. Camb. Phil. Soc., XI., p. 286, 1902.

$$a = 0.18, \quad b = 11,000;$$

and for Cell VII.,

$$a = 0.01, \quad b = 10,000.$$

The dotted curve in Fig. 4 is the curve obtained by using the former of the two sets of constants in the above theoretical equation. The close agreement between these curves and the theoretical curves indicates that the current is due to an emission of positive particles from the surface of the alkali metals. Dr. S. H. Anderson, working in this labora-

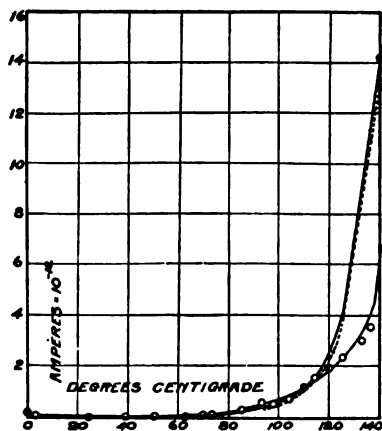


Fig. 4.

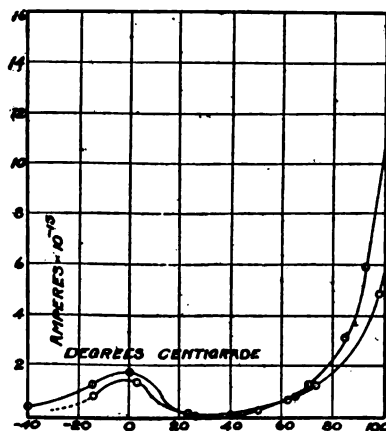


Fig. 5.

tory, has shown in a paper which will appear later that the same curves are obtained from the alkali metals *cæsium* and *potassium*.

However, at the temperatures below room temperature the above equation does not hold. A maximum value was obtained at about 0°C . as is shown by the curves in Fig. 5. As the temperature was still further decreased the magnitude of the current decreased and at the temperature of liquid air it was too small to be measured with any degree of accuracy. It will be seen that the curves in Fig. 5 are the same as those in Fig. 4 drawn on a different scale so as to show the small maximum at 0°C . It was observed that even with the best vacuum possible, the surface of the sodium-potassium alloy would not remain pure if an aluminium electrode were used. After a few weeks a white film would appear which caused a decrease in the maximum potential and current obtained from the cell. Upon examining several cells it was noticed that only those which had been left short-circuited for some time in the dark showed this effect. If care were taken not to leave the cell in the dark while short-circuited, the surface would remain pure for a long time. In the later cells investigated the aluminium electrode was replaced by one of

platinum. After four months these cells still show a perfectly pure alkali-metal surface. That is, a small beam of light is so reflected from the metal that one cannot see the spot where it strikes the surface. Time has not permitted a further investigation to determine whether this formation of a film was due to the gases contained in the aluminium electrode or to some action between the alkali vapor and this electrode.

No attempt has been made to explain all the phenomena described here. However, all the results point to the conclusion that some sort of positively charged particles are given off by the alkali metals in a high vacuum. As Dr. Anderson in a paper which is to appear later has shown that at the high temperatures this emission of positive electricity is so great as to counterbalance the ordinary photo-electric effect in the light, we conclude that this action is taking place in the light as well as when screened from light but is overbalanced by the emission of negatively charged electrons at ordinary temperatures in the latter case.

If we consider the current given by these cells as due to positive particles emitted from the surface of the metal, the number leaving the surface can be computed. For we have that

$$n = \frac{i}{e},$$

where n is the number leaving the surface per second, i is the current, and e is the charge on each carrier. At a temperature of 25° C. the current was found to be

$$i = 1.9 \times 10^{-15} \text{ E.M.U.}$$

Taking e to be 1.5×10^{-20} , this gives for the number leaving the surface per second,

$$n = \frac{1.9 \times 10^{-15}}{1.5 \times 10^{-20}} = 1.3 \times 10^5.$$

It is interesting to compare this with the total number of atoms in the surface layer of the alkali metal. An approximate value of the average area of cross-section of the atom in the alloy used is 1.7×10^{-15} sq. cm. The area of the surface was 8.4 sq. cm. so that the total number in the surface layer was

$$\frac{8.4}{1.7 \times 10^{-15}} = 5 \times 10^{15}.$$

Hence, the ratio of the total number of atoms in the surface layer to the number of particles leaving this surface per second is

$$\frac{5 \times 10^{15}}{1.3 \times 10^5} = 3.8 \times 10^{10}.$$

That is, if the current is considered as due to positive particles shot out from the metal, only one atom out of 3.8×10^{10} of those on the surface layer would be required to emit a positive particle per second in order to account for the total current. At 120°C . the current was 2.6×10^{-13} E.M.U., and consequently,

$$n = 1.7 \times 10^7.$$

The ratio of the total number of atoms in the surface layer of positive particles emitted per second would be

$$\frac{5 \times 10^{15}}{1.7 \times 10^7} = 3 \times 10^8.$$

These results seem to indicate that only a very small proportion of the atoms are active and that the number increases with the temperature. If it can be proved that this current is due to positive particles shot off from the active atoms (or molecules), it will give a direct proof of the theory of radiation which assumes that only a small proportion of the molecules of a radiating body are active.

These investigations are to be continued further in order to investigate more thoroughly the nature of this positive emission and to determine the source of the energy. The present investigation has been carried out during the past year under the direction of Dr. Kunz, whom the author wishes to thank most heartily for his many helpful suggestions and for his never-failing interest and aid in the progress of the work.

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DIFFRACTION GRATING METHOD FOR DETERMINING INDEX OF REFRACTION.

BY FREDERICK A. OSBORN AND HORACE H. LESTER.

THE method here described for determining the index of refraction of a liquid is believed to be new. The index of refraction of the liquid is determined by finding the ratio of the wave-lengths in air and in the liquid, by means of a Rowland plane grating, using an Abbe spectrometer. The equation for determining λ is given by

$$m\lambda = 2d \sin \theta,$$

and for finding μ , if d is constant,

$$\mu = \frac{\lambda_a}{\lambda_w} = \frac{\sin \theta_a}{\sin \theta_w},$$

where the subscripts refer to the air and liquid values respectively. But the value of d changes with the temperature because of the thermal expansion of the speculum metal of the grating.

Let t_1 be the change in the temperature of the grating for the air readings, from the temperature when d_0 was determined;

t_{11} , the corresponding change in temperature for the liquid readings;

α , the coefficient of expansion of the grating.

Then

$$\begin{aligned} \mu &= \frac{d_0(1 + \alpha t_1) \sin \theta_a}{d_0(1 + \alpha t_{11}) \sin \theta_w} = \frac{\sin \theta_a}{\sin \theta_w} \left[1 + \alpha \frac{(t_1 - t_{11})}{1 + \alpha t_{11}} \right] \\ &= \frac{\sin \theta_a}{\sin \theta_w} [1 + \alpha(t_1 - t_{11})] \text{ approx.} \end{aligned}$$

Since t_{11} does not differ much from t_1 and $t_{11} \alpha$ is small in comparison to unity.

APPARATUS AND METHOD.

In the Abbe spectrometer (Fig. 1) the functions of the collimator and observing telescope are united. The plane grating, mounted on the rotating table, receives the light from the objective and returns the diffracted light along the same path as the incident light, so that it strikes the plane side of the objective lens, normally, thus avoiding any correction for refraction at the objective.

A Rowland grating of approximately 15,000 lines to the inch was used. The grating was mounted on the end of the movable spindle, Fig. 2. A double-walled metal tank was mounted on the fixed circle of the spectrometer as shown, so the grating could be completely immersed in the liquid, which was distilled water. A heating coil was wound on the outside of the tank and then the whole was wrapped with several layers of asbestos. Water, placed in the space between the double walls, helped to hold the temperature in the grating chamber more nearly constant. The work was done near 20°C .

The temperature of the inner chamber was obtained by a thermometer reading directly to $1/10^{\circ}\text{C}$., which had been recently calibrated by the Bureau of Standards. A stirrer, motor driven, kept the liquid well agitated.

A preliminary investigation with the aid of thermo-elements made certain that the thermometer indicated the true temperature of the liquid throughout the bath as well as the temperature of the grating with an error of less than $.02^{\circ}\text{C}$.

The lines used were those due to sodium, hydrogen and mercury,

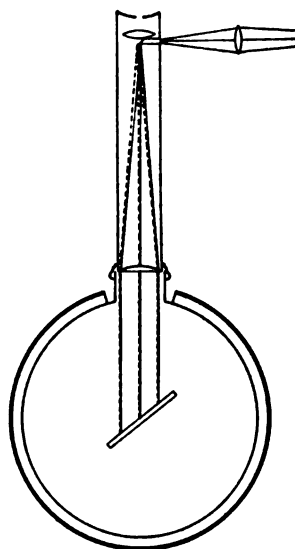


Fig. 1.

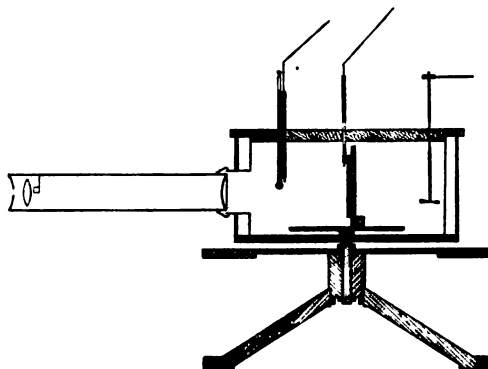


Fig. 2.

giving values for μ well distributed throughout the spectrum. The second, third or fourth order spectrum was used, depending on the brightness of the given lines.

The readings in air and in water for the various lines were made as in the usual grating method for finding μ .

Table I. gives the results as obtained by the authors at different times, working separately and together.

TABLE I.

Line.	t_1	θ_a	t_{11}	θ_w	μ_t	Correc- tions.	μ_{20°	Experi- menter.	Order of Spectrum.
H 656	19.5°	35°38'18"	20°	25°57'30"	1.33115	-1	1.33114	Lester	3d
	19.7	18	19.8	27	18	-2	16	Lester	3d
	19.8	17	20°	30	14	0	13	Lester	3d
	20.2	17	20°	30	14	0	14	Lester	3d
	20.3	16	19.9	29	14	-1	13	Lester	3d
	20.5	15	19.6	24	20	-4	16	Lester	3d
							Mean 1.33114-		
Na 5,896	20°	20°25'34".7	20	15°16'41".5	1.33299	0	1.33299	Lester	2d
	20°	35		40	302	0		Lester	Apr. 6
		35		40	302	0		Lester	to
		36		40	303	0		Lester	Apr. 26
		35		40.7	302	0		Lester	
		34.7		41	299	0		Lester	
		35.5		40	299	0		Lester	
		34.7		40	301	0		Lester	
		35		40.2	302				
							Mean 1.33301		
Na 5,890	18	31°31'55"	20.4	23°5'53"	1.33305	-5 } +4 }	1.33304	Lester	3d
	20	20°24'14"	20	15°9'40"	04	0	304	Lester	2d
	20	16	20	39	06	0	06	Lester	
	20	14	20	41	03	0	03		
	20	14	20	40	04	0	04		
	20	14	20	41	03	0	03		
	19.2		20.3	39	02	-1 +3	04		
							Mean 1.33304		
Na 5,893							1.333025		

Line.	t_1	θ_a	t_{11}	θ_{∞}	μ_t	Correc- tions.	μ_{30}	Experi- menter.	Order of Spectrum.
Hg 579	20.3	30°56'22"	20.1	22°40'55"	1.33328	+1	1.33329	Osborn	3d
	20.2	25	20.1	55	32	1	33	Osborn	3d
	20.3	23	20.1	53	35	1	36	L.	3d
	20	23	20.1	52	33	1	34	L.	3d
	20	23	20.1	52	33	1	34	O. & L.	3d
	20	21	20.1	52	32	1	33	O. & L.	3d
							Mean 1.33333		Dec. 26-29
Hg 577	20	30°48'54"	20.1	22°35'34"	1.33340	+1	1.33341	O.	3d
	20	53	20.1	34	40	1	41	O.	3d
	20	53	20.1	32	43	1	44	L.	3d
	20	52	20.1	33	41	1	42	L.	Dec. 28-30
	20	53	20.1	34	40	1	41	O. & L.	
	20	53	20.1	34	40	1	41	O. & L.	
							Mean 1.333416		

Line.	t_1	θ_a	t_{11}	θ_{∞}	μ_t	Correc- tions.	μ_{30}	Experi- menter.	Order of Spectrum.
Hg 546	20.8	29° 0' 7"	20.5	21°18'14"	1.33439	5	1.33444	S. & M.	3d
	19.9	29 0 9	19.8	13	50	1	51	S. & M.	
	20.0	29 0 8	20	16	49	0	49	O.	
	20.1	29 0 7	20	17	43	0	43	O.	
	19.2	28 59 59	19.8	10	48	3	45	L.	
	19.4	29 0 1	19.8	8	52	2	50		
	19.2	29 0 1	19.8	8	51	3	48		
	20.0	29 0 0	20.8	15	40	8	48		
							Mean 1.33447		
H 486	22.7	25°34' 7"	21.9	18°50' 3"	1.33689	20	1.33712	L.	3d
	22.6	8	21.9	2	93	21	17	L.	
	22.7	16 43 18	21.9	12 25 41	92	21	13	L.	
	22.6	17	22.1	40	93	22	15	L.	2d

Line.	t_1	θ_a	t_{11}	θ_ω	μ_t	Correc- tions.	μ_{20}	Experi- menter.	Order of Spectrum.
Hg 436	20	31° 3'45''	20.1	22°38'30''	1.34031	-1	1.34032	O.	4th
	20	44	19.9	28	33	-1	32	O.	4th
	20	43	20.1	30	26	-1	27	L.	4th
	20	39	19.9	28	30	-1	29	L.	4th
	20	43	19.9	29	29	-1	30	O. & L.	Dec. 26-29
	20	44	20.1	29	31	-1	30	O. & L.	
		44	19.9	28	29	-1	30		
	20.1	22 45 57	20	15 46 54	28	0	28	L.	3d
	20.1	58	20	53	30	0	30	L.	March
	20.1	57	20	52	29	0	29	L.	March
	20.1	57	20	54	28	0	28	L.	March
							Mean 1.340295		

In reducing our values from μ_t to μ_{20} the empirical equation of Flatow, ($\mu_t = \mu_0 - 10^{-8}(at + bt^2 + ct^3 + dt^4)$) was used, where a, b, c, d are the constants, differing for each wave-length. Martens¹ has calculated the values of du/dt for $t = 20^\circ$ and for λ varying from .000214 to .000589. From these values a curve was plotted and the corrections to be applied to our data were taken directly from the curve. The correction due to the expansion of the grating ($\alpha = .000019$) amounts to 2×10^{-8} for a temperature difference of 0.1° C.

ACCURACY OF THE METHOD.

θ_a and θ_ω were found to $\pm 1.5''$ and this gives an error in μ of $\pm 2 \times 10^{-5}$. The temperature of the grating was known to $\pm .02^\circ$ C. and an error of this amount affects μ by $\pm 2 \times 10^{-7}$. The temperature of the bath was not in error by $.05^\circ$ C. and this would change μ by $\pm 5 \times 10^{-6}$. The probable error in μ is therefore $\pm 2 \times 10^{-5}$.

Table II. gives the various values for μ_{20} for the wave-lengths worked with, as found by other workers and other methods,² and those obtained by our new method. The curve, Fig. 3, which is drawn from the mean values for μ_{20} obtained from the values found by the different workers, shows the variation in their values from the mean and also the position on the mean curve of our values.

SUMMARY.

1. A new method for finding μ for some liquids has been found.
2. The method is simple in theory and practice.

¹ Landolt-Bornstein, Phys.-Chem. Tab., 1905, pp. 669-670.

² Landolt-Bornstein, Phys.-Chem. Tab., 1905, pp. 669-670.

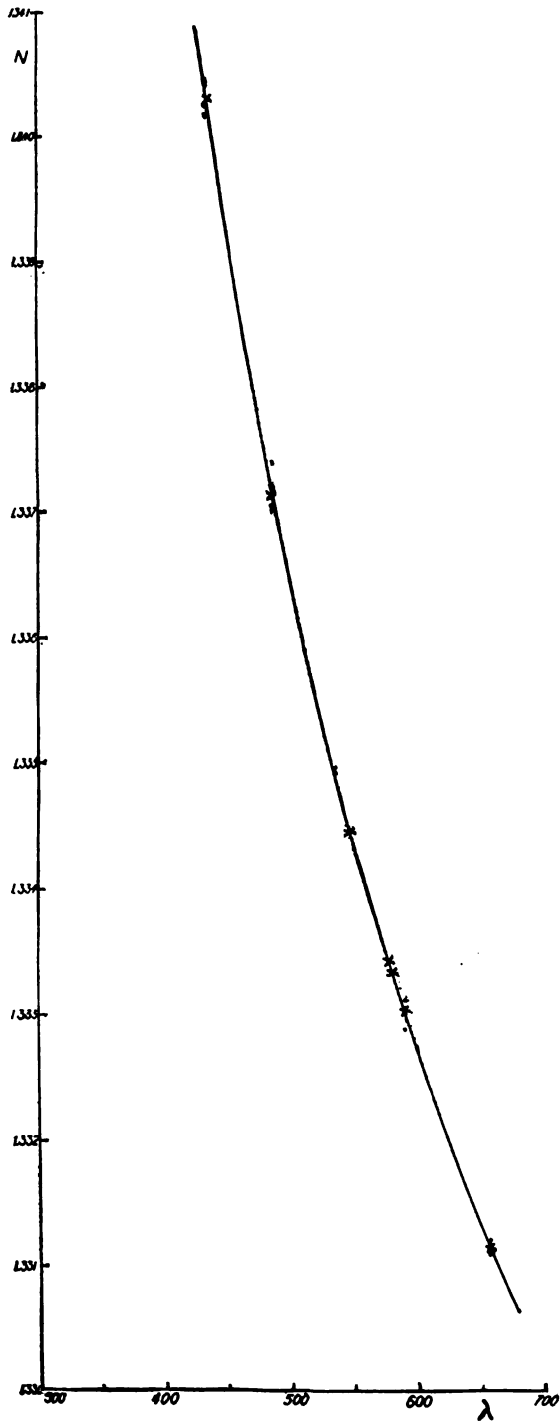


Fig. 3.

Water. Values from new method of $dN/d\lambda$ are indicated by crosses; values from other workers are indicated by dots.

TABLE II.
Index of Refraction of Water at 20° C.

Experimenter.	λ			
	$H_{\gamma_{486}}$	$H_{B_{486}}$	$N_{a_{486}}$	$H_{a_{486}}$
Dufet.....	1.34015	1.33701	1.33292	1.33109
Bender.....	23	05	87	100
Schutt.....	38	15	300	116
Bruhl.....	44	19	04	19
Wullner.....	36	14	—	21
Damien.....	35	05	—	08
Simon.....	45	12	306	08
Landolt.....	38	12	295	—
Ruhlman.....	—	—	301	—
Lorenz.....	—	—	300	—
Jamen.....	—	—	299	113
Walter.....	—	—	300	—
	Mean	Mean	Mean	Mean
	1.34032	1.33711	1.332994	1.33113
	$H_{g_{486}}$			
Osborn and Lester.....	1.340295	1.33714	1.33302	1.33114

3. Its accuracy compares favorably with any of the present methods.
4. By making the liquid chamber smaller and bringing the telescope nearer the grating, smaller amounts of the liquids would be required.
5. Any liquid not attacking the grating could be used, as the alcohols and most of the organic liquids.

PHYSICS LABORATORY,
UNIVERSITY OF WASHINGTON, 1912.

A STUDY OF THE SPHEROIDAL STATE.

BY C. W. BATDORF.

JOHANN GOTTLÖB LEIDENFROST, in a work on surface tension published in 1756, was probably the first to publish accounts of the behavior of a liquid on a hot, smooth plate. From this it was known as Leidenfrost's phenomenon, but later the term "spheroidal state" was applied. Dr. Berger¹ gives a summary of the extensive investigations of the German, French and English physicists, together with many references and some new work. Experiments were carried on with a great number of liquids and mixtures, most of the work being done to discover whether the liquid touches the hot plate or not, and if not, why. By 1863 the leading theory was the present one, that the drop rests on a cushion of its own vapor. Berger and Tyndall supported this theory, the latter calling attention to Mr. George Stephenson's idea of having his locomotive boilers rest on a cushion of steam.² Another theory claimed that the heat pressure thrusts the liquid back; and another, that the heat so weakens the adhesion that the cohesion predominates. For plates Berger used hemispherical shells of copper, silver and platinum, and molten metals. He even experimented upon leaves, fruit parings, and grasses, which twirled rapidly on the hot plate. He concluded that any substance which can be liquefied can pass into the spheroidal state if the plate can be sufficiently heated. The early investigators also tried to account for the existence of the starforms. Schnauss³ published drawings of the starforms; Seyffer⁴ improved the figures by bowing; and both men likened the starforms to sound figures. No agreement could be reached for the temperature of water in the spheroidal state. Mercury-glass thermometers were used, and Döbereiner, Marchand, Laurent, and Person obtained results ranging from 82° C. to 101° C. Boutigny found the temperature to remain constant at 96.5° C., while Berger found the temperature to range from 96° C. to 98° C.

To determine the temperature of distilled water in the spheroidal state I used a platinum-platinum rhodium thermo-couple connected to

¹ Pogg. Ann., 1863, Vol. 119, page 594.

² Phil. Mag., 4, 10, page 353.

³ Pogg. Ann., 79, page 432.

⁴ Pogg. Ann., 80, page 578.

a sensitive galvanometer, the latter being read with telescope and scale. By placing the disc-shaped weld of the thermocouple in a horizontal position the temperature of a small drop could be determined. Temperature readings, taken as rapidly as an observer could record them from the telescope, show that the temperature fluctuates constantly. Starting with a mass of about two grams on a level iron plate the following temperatures were indicated:

95°.66 C.	95°.66 C.	94°.55 C.	95°.10 C.
97°.50 C.	94°.68 C.	95°.10 C.	93°.56 C.
96°.55 C.	94°.42 C.	94°.96 C.	91°.76 C.

With the couple in boiling water there was no motion of the galvanometer mirror. In the above experiment the mirror was motionless for only a small fraction of a second at a time. The table shows that the temperature, although fluctuating, gradually decreased as the globule of water became smaller from evaporation. Using a weld about 2.5 mm. in diameter, 78° C. was the lowest dependable temperature observed. Occasionally the range of temperature fluctuation was very small for a minute or more, as may be seen from the following table:

96°.09 C.	96°.17 C.	96°.24 C.
96°.09 C.	96°.09 C.	96°.38 C.
96°.24 C.	95°.96 C.	96°.38 C.

This shows an approximately equilibrium temperature of 96°.18 C., a mean not far from Boutigny's value of 96°.5 C. But ordinarily the temperature fluctuated over a range of about 6° C. The temperature will vary also with the atmospheric pressure.

Much attention has been given to the minimum temperature of the plate at which water will assume the spheroidal state. To determine this two plates were used, a level iron plate 2.5 cm. thick and 5 cm. in diameter, and a brass plate of about the same dimensions with an inverted cone-shaped surface of large angle. A shield with a chimney disposed of the heated gases. A small hole bored into the plate and filled with mercury, to contain the couple, furnished a means of determining temperatures below 350° C. A drop of about 2 grams of water on the iron plate, in equilibrium in the spheroidal state while the plate was cooling, touched the metal when it had cooled to 260° C. Under analogous conditions a drop of about 0.5 gm. remained in the spheroidal state until the plate cooled to 230° C., a range of 30° C. due to using different-sized drops. A small drop of water on the cooling brass plate touched the plate very suddenly at 228°.4 C. A still smaller drop on the same plate all disappeared by slow evaporation, the temperature of the plate being 172°.9

C. at the close of the action.¹ In considering the "minimum temperature of the plate," the conductivity of the material of the plate must be considered. A silver plate will doubtless maintain a given spheroid of water when at a lower temperature than an iron plate of the same dimensions. It might be shown by using a level plate one half of one metal and the other half of the other metal. Further, the minimum temperature will vary with the pressure. Buddle found that in an exhausted receiver water passed into the spheroidal state even when the temperature of the support he used was not more than 90°C .² The "minimum temperature of the plate" is therefore rather an indefinite conception unless accompanied by specifications as to its material and dimensions, the size of the drop, and the atmospheric pressure.

The most extraordinary phenomena of the spheroidal state of liquids are the starforms and other regular figures. They may be observed on a level, or better, on a hollow cone-shaped plate, if the temperature is above 450°C . A globule of water resting quietly on such a plate will soon begin to vibrate radially in the form of a star with an even number of curved points. These figures must be observed to be appreciated. A $2n$ -point star consists of an n -point star occupying at one instant a given position, and at the next instant occupying a position making an angle π/n with its first position, with the same center. The oscillations from one of these positions to the other are strictly radial, so that the $2n$ -point star owes its appearance to the persistence of the superposed retinal images.

A vibrating $2n$ -point star becomes an n -point star standing perfectly still when viewed through the stroboscope, properly regulated as to speed. The stroboscope disc should be very light so that the speed can be quickly regulated by a sliding rheostat in the motor circuit. From the speed of the disc and the number of apertures the vibration rate of the star can be calculated. The vibration rate is too low to observe the details of the action in a rotating mirror; but the stroboscope shows that, while the complete star is a gem of perfection and symmetry, the parts of the star are sometimes asymmetrical. This is particularly true of the 5-point star vibrating to compose the 10-point star.

For a given vibrating globule the speed of the disc must be steadily increased in order to keep the n -point star in view; because, with certain

¹ The thermo-couple used formed a highly sensitive thermometric system. In the above experiments, when the water touched the metal plate the galvanometer mirror jumped instantly and moved so rapidly that the eye could not follow the scale readings. This shows at once the rapidity of heat conduction through the mercury and the plate and the sensitiveness of the system.

² Ganot's Physics, 17th ed., page 385.

limitations, the vibration rate increases as the globule decreases by evaporation. A globule of water of the right magnitude will vibrate as a 10-point star with ten complete vibrations per second. The points are prominent and far apart, the amplitude of vibration is near the maximum and the vibration rate is near the minimum. As the drop evaporates the circumference decreases, the star points are crowded closer together, their curvatures and the vibration rate increase, while the amplitude decreases. This action continues until the vibration rate reaches a value of from 18 to 24 complete vibrations per second. Then the vibrations gradually die away, and the drop remains for a few seconds in comparative equilibrium, or with rapid vibrations of small amplitude. The drop soon begins to vibrate again, this time as an 8-point star, but with ten complete vibrations per second as before. The points are large, and the amplitude has a maximum value. As the circumference decreases, owing to evaporation, the points are crowded closer together, as before. As before the curvature of the points and the vibration rate increase, while the amplitude decreases. When the rate reaches a value of from 18 to 24 vibrations per second the drop once more regains equilibrium, only to begin vibrating again as a 6-point star, with a rate of about ten vibrations per second.¹ During the passage from a $2n$ -point star to a $(2n-2)$ -point star the drop usually vibrates at a high rate with small amplitude, forming other regular figures of exquisite pattern.

The number of star points is not strictly determined by the circumference of the drop, as is shown by accompanying photographs, Figs. 10 and 11. This case shows two drops of water (made milky with alcohol and rosin) of approximately the same size, under identical conditions. Yet one vibrates with ten points and the other vibrates with sixteen points. Various interesting modifications can be made in the figures by the insertion of fine wire circles, squares, and stars, the vibrations being quite stable. Depressing the drop increases the circumference and the amplitude, and decreases the rate. Raising the wire frame will lift part of the drop, producing the opposite result. If a little water be allowed to trickle down the wire stem to the middle of the vibrating drop, the rate may be decreased considerably below ten vibrations per second, without changing the number of points.

The 4-point star is composed of an ellipsoid vibrating radially in two positions, not always at right angles with each other. The vibration rate varies from less than ten to more than forty complete vibrations per second. The two rotations—one more or less rapid about a vertical

¹ The numbers relating to maximum and minimum vibration rates are illustrative only, as they vary over a wide range, and are easily subject to control.

axis which interferes with photographing, and the other a slow rotation of a globule equilibrium about a nearly horizontal axis—have been described by Berger. The stationary, concentric circular waves in the surface of the drop, just preceding the starform, also have been mentioned. Often a large globule will vibrate elliptically as a whole, while vibrating at the same time in small points around the perimeter of the ellipse. The vibrating starform gives rise to a distinct sputtering sound.

Various mixtures and solutions, both of salts and of liquids, of varying degrees of concentration, furnish an endless variety of modifications in the figures. A deposit of salt may be removed from the plate by dropping cold water on to the plate from some little height. The fact that salt is deposited and may be removed as indicated helps to show the nature of the contact. In mixtures like oil and water the water becomes encased in the hot oil and “explodes” so that observation is not possible.

The starforms and other figures are readily produced by other liquids, such as alcohol, kerosene, paraffine, benzine, turpentine, and ether. A liquid differing from water in density and in surface tension will generally differ also in the starform and in the vibration rate. In the liquids named the star points are smaller than in water. The vapor of a combustible liquid may be ignited without interfering with the star. This method of evaporating combustible liquids has some advantages. It shows, for example, that the residue of paraffine wax is practically an explosive.

For liquid air the star points are smaller and the vibration rate is higher than for any other liquid tested. The plate should be a hollow cone whose angle is less than 160° to keep bubbles from destroying the star. The sounds produced by liquid air are much louder than the sounds produced by water. As the drop grows smaller the vibration rate increases, until a sound corresponding to 500 or more complete vibrations per second is produced. Ice formed from the vapor in the air interferes with the vibrations of a small drop and prevents the production of a very shrill sound. A globule of liquid air, dropped on the surface of water or other liquid, immediately forms a star and soon freezes the surface of the depression in which it rests. A block of ice is unsuitable for a plate because it cracks badly. On sulphuric acid liquid air rotates rapidly, sometimes in the form of an annular ring. Ether will form a star for a short time on the surface of warm water. Liquid air dropped on to a level plate, at the room temperature, from a height of 10 or 15 cm., is broken up into a number of small drops, free from interfering ice particles. The impact sets these small drops into violent vibration in the ellipsoidal form described for a small drop of water. These figures are of remarkable beauty.

A study of the forces acting upon the globule, and of the condition of the contact, are the first steps in understanding this interesting action of liquids. The chief forces present are weight, and the upward thrust of the steam, acting vertically in opposite directions; and surface tension, acting tangentially to the surface at any point. The globule rests upon its own vapor; that is, upon a substance to which it does not adhere, so that surface tension is able to form the liquid into a "spheroid." A globule on such a springy cushion, subjected to the action of these three forces, can as well be expected to form a regular as an irregular figure. Since surface tension is active in a cold liquid also, therefore, if an adequate substitution can be made for the other conditions it should be possible to secure a starform in a cold liquid. A simple experiment will show that it is possible. A globule of water placed upon a nasturtium or a pond lily leaf, so that the horizontal section is approximately a circle, subjected to vertical mechanical vibration of suitable frequency and amplitude, will vibrate as a star, similar to one formed on a hot plate and quite as perfect in form. A dessert spoon, blackened in the flame of a candle, makes an excellent surface for water; and the curved handle held firmly on the table while the spoon projects over the edge can be used to make the spoon vibrate forming star-shaped figures. This can be done with any liquid for which a suitable surface can be obtained.

Since vertical vibration, combined with surface tension, is sufficient to account for the starforms, it is clear why such figures do not appear on the standard perforated silver plate commonly used in laboratories. The vapor escapes through the perforations and cannot lift the globule; hence one force is wanting.

In studying the vibrations of a cold spheroid, high temperatures do not then interfere, evaporation is negligible, and rotation is lessened or eliminated, which makes photographing easier. The vibration rate and the amplitude may be controlled, and an endless variety of figures can be produced. In such a study a large, electrically driven fork, horizontally mounted, is suitable. Each prong may have an adjustable load to vary the rate of vibration. Of the two forks used by the writer one had a frequency of 64 and the other of 128 complete vibrations per second, and the loads were 50 grams each. A thin spring board may be clamped above the fork, and the vibrations may be communicated to it by a vertical rod. The vibrating surface (for water a watch glass covered with the soot of a candle flame) may be fastened to the board by rubber bands. A sliding rheostat in circuit will aid in adjusting the amplitude.

Some of the best studies can be made with fairly clean mercury on a watch glass, the surface of which has been tarnished with oxidized

mercury to prevent too great adhesion between the mercury and glass. This last condition is important. When the fork is in unison with a natural period of the mercury globule a small amplitude will suffice to form a perfect star. When the fork is not in unison with a natural period, still the globule often passes through a cycle, taking on several perfectly symmetrical forms in rapid succession. A particularly rich variety can be obtained by adjusting the loads at different distances on the prongs and allowing the vibrations to decrease slowly, with the circuit broken.

The surface of vibrating mercury often shows a pattern of concentric circular stationary rings, whose number and radii vary with the period. The number of points in the mercury star also depends on the period. Using the large fork described above, with the prongs loaded, a mercury globule about 25 mm. in diameter vibrated with eight points. With the load removed the globule vibrated with twenty points. There were three rings in each case, but corresponding radii were different in the two cases. Upon using the smaller fork, the same globule vibrated with many points, perhaps thirty-four, with six rings. Thus, a given globule has many natural periods, depending upon the figure produced.

By holding a stretched string horizontally so that its image is seen in the surface of the globule near the region of the outer circle, where the amplitude is very small, the reflected image will be seen to have the same form as the star-points in the perimeter, giving a hint as to the character of the surface vibrations.

A large drop vibrating as a 20-point star with three rings had 20 radii, one extending to each point. Light from an incandescent bulb was reflected from each area bounded by two arcs and two radii. Similar radii and rings sometimes appear in a spheroid of water. Both cases strikingly resemble Chladni disc patterns.

An inexhaustible variety of mercury figures may be formed, depending upon the size of the drop, the amplitude and rate of vibration, and other conditions. Pouring water over the mercury globule again furnishes new forms. A few of these oddities are shown in the accompanying drawings.

A star formed on a hot plate has only an even number of points, all of the same size. One characteristic mercury figure might be considered a star with an odd number of points. Such a star, with seven points, is drawn to scale in Fig. 1. The outside perimeter represents the extreme position. The dotted line is the perimeter in the other position. The points *a, b, c, . . .* are stationary, while the star-points *d, f, . . .* vibrate to *e, g, . . .* and back again. Hence the globule must increase and decrease in thickness, a movement quite different from that of the ordinary mercury star shown in Fig. 2. Fig. 2 is a characteristic mercury star with 12 points, 12 radii and one ring. The solid line is the star

in one position, the dotted line is the star in the other position. Fig. 3 is a form of frequent occurrence. It is the more unexpected when it is remembered that the horizontal section of the watch glass is a circle and that Figs. 1 and 2 occur in the same glass. Fig. 4 is cold water vibrating as an 8-point star on a nasturtium leaf. All figures are drawn to scale.

The starforms can be photographed, using an arc light and a camera with short focus. Figs. 5 to 12 are taken from photographs. Fig. 5 is a

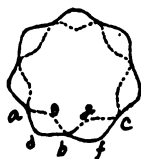


Fig. 1.

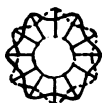


Fig. 2.

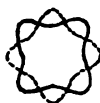


Fig. 3.



Fig. 4.

12-point mercury star on a white background, exposure half a second. The curved points are shown, except on the side next the arc. Fig. 6 is a 14-point, Fig. 7 a 16-point mercury star, both on a dark background, exposure one second. Fig. 8 is a star with 24 points, formed by placing a watch glass on top of the mercury globule. Fig. 9 is a 4-point ellipsoidal star, taken from water made milky with rosin and alcohol. Reference has already been made to Figs. 10 and 11. If the water is made translucent, the entire outline of two n -point stars may be photographed. Fig. 12 is such a photograph for an 8-point star. At the base of any point can be seen the concave part of the perimeter between its two neighboring points. The focal distance was 25 cm.

The movements of fine particles suspended in a liquid in the spheroidal state show the drop to be in a state of violent internal agitation, which may help account for the rapid fluctuations in temperature. The ice particles in liquid air collect in a circle around the edge of the drop and are there given a rapid rolling motion. A globule mechanically vibrated shows equally rapid internal motions.

Water dropped on to a level hot plate breaks up into a number of small drops, each vibrating rapidly with considerable amplitude. As these small drops dart about on the hot plate they often come violently into contact with each other. Two such drops do not coalesce, but fly apart like two rubber balls. As the amplitude decreases, even though the vibration rate increases, the drops are able to coalesce when they meet. The same is true for liquid air on a level plate at the room temperature; and thus the ellipsoids already described can be viewed for some time. Small spheres of mercury falling on to the surface of vibrating mercury often glide over the surface for a time. It is difficult to pick up a vibrating mercury globule in a glass dropper; the mercury runs

away from the dropper. A vibrating mass of liquid seems, on account of its motion, to be able to maintain an independent existence in much the same way that a vortex ring, or other moving system, maintains its existence.

A small stream of water dropping from a height of about 30 cm. on to



Fig. 5.



Fig. 8.

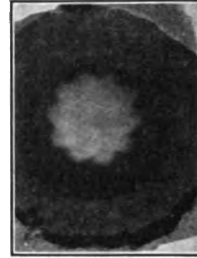


Fig. 10.

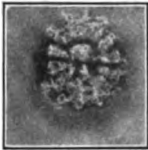


Fig. 6.



Fig. 11.



Fig. 7.



Fig. 9.



Fig. 12.

the surface of quiet water will, from the impact, produce a large number of small spherical globules which float for a few moments on the surface.¹ The globules are projected rapidly outward from the source of formation, and they can be seen to be in rapid vibration. They glide over the surface for some distance, and then, when their motion of translation becomes slow, they coalesce with the main body of liquid. A tiny column of water is splashed upward about half a centimeter, as described in Worthington's "Splashes." It can be seen with the eye on the level of the water. By the same method each sphere can be seen to float

¹ Poynting and Thompson, *Heat*, p. 184.

in a shallow basin and when the light is right a train of ripples can be seen in front of a moving globule. These globules are not in contact with the surface of the water, for instead of following the water down the overflow edge of the vessel, they jump off, clearing the wall by some distance. There they may be caught on a plate of glass coated with paraffine. A globule 3 mm. in diameter is a fairly large one. They can be formed on distilled water, but not on oily or contaminated water.

Occasionally two globules will unite and continue as one. Frequently a globule will coalesce or partly coalesce with the liquid, leaving a smaller globule in its place. The drops may have one or more rotations but they do not roll, since there is no proper contact. The motions of vibration, translation and rotation, together with surface tension, are probably sufficient to account for their existence.

If a light, cylindrical glass jar containing wood alcohol be bowed, myriads of spherical globules will be thrown toward the center by the vibrating segments.¹ There they continue to float for a few moments, repelling one another when they come in contact. A close inspection shows that they are in rapid vibration, which accounts for their ability to remain on the surface.

It is impossible for a liquid in the spheroidal state to boil, since the lower side is free to evaporate into the atmosphere.

Water boiled in a smooth vessel gives rise to superheating effects, with explosive boiling. Something of this sort might be expected to take place in a liquid globule in the spheroidal state; but as a matter of actual experience, after reaching its equilibrium temperature, just below its boiling point, the liquid is able to lose heat by radiation and evaporation as rapidly as it receives heat by radiation from the plate, and by conduction from its cushion of steam.

An incandescent silver ball suspended in a vessel of boiling hot water will not keep the water at that temperature. Even the rising column of vapor is below the boiling point temperature. A thermometer held with its bulb at the side of the glowing silver, at a distance of 3 cm. from it, indicates a falling temperature. At this point the temperature will drop at least 5°.0 below boiling.

A bunsen flame directed against the surface of a beaker of boiling hot water will not keep the water at that temperature. Hence, by means of radiant energy alone, directed against one side of a liquid, aided by the contact of hot gases, the liquid cannot be made to boil, nor be brought to its boiling temperature.

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¹ Poynting and Thomson, *Heat*, p. 184.

A THEORETICAL CORRECTION TO THE DROP METHOD OF DETERMINING THE ELEMENTARY CHARGE.

BY ARTHUR C. LUNN.

THE absolute determination of the elementary charge by the use of drops of liquid as carriers, in the refined form developed by Millikan,¹ depends upon the measurement of the steady velocity of the drop through the surrounding medium under gravity, with and without the presence of an electrostatic field. The computation assumes that the coefficient of mobility of the drop is the same in both cases. If, however, the liquid of which the drop is composed has a dielectric constant different from that of the surrounding medium, the surface stresses produced by the field tend to distort the drop into a somewhat elongated form, up to the configuration at which they are equilibrated by corresponding inequalities in the normal component of the surface tension, due to variations in the curvature of the surface. This indicates that the field may be expected to modify the factor of mobility, implying a correction to the value of the charge as computed from the observed velocities. There is here obtained an approximate formula for the amount of this ellipticity, from which it appears that under the actual conditions of Millikan's measurements the correction is inappreciable.

Let $\mathbf{E}$, $\mathbf{D}$ be the electric field and displacement and w the energy concentration, then on a surface whose unit outward normal is $\mathbf{n}$ the vector of stress is

$$\mathbf{S} = (\mathbf{D}\mathbf{n})\mathbf{E} - w\mathbf{n}.$$

In the case of a sphere of uniform dielectric constant ϵ , placed in a field previously uniform, of strength F , there will be symmetry about the axis of the field, so that it is sufficient to consider the components of the vectors in a meridian plane, along and perpendicular to the radius, to be denoted here by subscripts 1, 2 respectively. Then inside the sphere, at the surface, may be put

$$\mathbf{E} = (E_1, E_2), \quad \mathbf{D} = \frac{\epsilon}{4\pi} (E_1, E_2), \quad \mathbf{n} = (-1, 0),$$

$$w = \frac{\epsilon}{8\pi} (E_1^2 + E_2^2), \quad S_1 = -\frac{\epsilon}{8\pi} (E_1^2 - E_2^2), \quad S_2 = -\frac{\epsilon}{4\pi} E_1 E_2$$

¹ PHYS. REV., 32, pp. 349-397, April, 1911.

and outside, because of the continuity of $\mathbf{E}$ tangentially and of $\mathbf{D}$ normally:

$$\mathbf{E}' = (\epsilon E_1, E_2), \quad \mathbf{D}' = \frac{1}{4\pi} (\epsilon E_1, E_2), \quad \mathbf{n}' = (1, 0),$$

$$w' = \frac{1}{8\pi} (\epsilon^2 E_1^2 + E_2^2), \quad S_1' = \frac{1}{8\pi} (\epsilon^2 E_1^2 - E_2^2), \quad S_2' = \frac{\epsilon}{4\pi} E_1 E_2.$$

Thus the resultant stress is entirely normal, of value

$$S_1 + S_1' = \frac{\epsilon - 1}{8\pi} (\epsilon E_1^2 + E_2^2).$$

Inside the sphere the field is still uniform, but of strength reduced in the ratio $3/(\epsilon + 2)$, so that

$$E_1 = \frac{3}{\epsilon + 2} F \cos \varphi, \quad E_2 = \frac{3}{\epsilon + 2} F \sin \varphi,$$

where φ is the polar angle reckoned from the axis of the field. Hence

$$(1) \quad S_1 + S_1' = \frac{9(\epsilon - 1)}{8\pi(\epsilon + 2)^2} \{(\epsilon - 1) \cos^2 \varphi + 1\} F^2.$$

Since this is greatest at the poles of the sphere, the tendency is to produce elongation until compensation is reached through a corresponding variation in the product of surface tension and total curvature of surface.

For a first approximation the deformed surface may be considered as a prolate ellipsoid of revolution, whose meridian curve, in terms of the eccentric angle t , is given by

$$x = a \cos t, \quad y = b \sin t.$$

The first principal radius of curvature of the surface, that of the meridian curve, is then

$$R = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y'' - x''y'} = \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{\frac{3}{2}}}{ab}$$

and the second principal radius is

$$R' = y \sec \psi = \frac{b \sin t}{a} \sqrt{a^2 + b^2 \cos^2 t},$$

where ψ is the angle between the y -axis and the normal to the ellipse, so that

$$\tan \psi = \frac{b \cot t}{a}.$$

Thus the total curvature of the surface, expressed in terms of a and the eccentricity e is

$$k = \frac{1}{R} + \frac{1}{R'} = \frac{1}{a} \cdot \frac{2 - e^2 - e^2 \cos^2 t}{(1 - e^2)^{\frac{1}{2}}(1 - e^2 \cos^2 t)^{\frac{1}{2}}},$$

and the corresponding normal pressure from the surface tension T , expanded in powers of e , is

$$(2) \quad P = kT = \frac{2T}{a} (1 + e^2 \cos^2 t + \dots).$$

At corresponding points the electric stress on the elongated surface will differ from that on the sphere by terms containing powers of the measure of deformation; similarly the angles t , φ differ by terms containing powers of e^2 . Hence the condition of equilibrium in terms of lowest order will be obtained by equating the coefficient of $\cos^2 \varphi$ in (1) to the coefficient of $\cos^2 t$ in (2), so that the eccentricity is determined by

$$(3) \quad e^2 = \frac{9}{16\pi} \left(\frac{\epsilon - 1}{\epsilon + 2} \right)^2 \frac{aF^2}{T}.$$

Under the same conditions as for the ordinary form of Stokes' law for a sphere, moving steadily through a viscous medium, the resistance to the ellipsoid¹ is the same as to a sphere of radius

$$r' = \frac{8}{3} \frac{ab^3}{\chi + \alpha a^2},$$

where

$$\alpha = ab^3 \int_0^\infty \frac{ds}{(a^2 + s)\Delta}, \quad \chi = ab^3 \int_0^\infty \frac{ds}{\Delta},$$

$$\Delta^2 = (a^2 + s)(b^2 + s)^2.$$

The radius of the sphere of equal volume is however

$$r = (ab^3)^{\frac{1}{3}}.$$

If then the material of the drop be regarded as incompressible, the effect of the elongation of form is to diminish the resistance in the ratio

$$(4) \quad \frac{r'}{r} = \frac{4e^2}{3(1 - e^2)^{\frac{1}{2}} \left\{ \frac{1 + e^2}{2e} \log \frac{1 + e}{1 - e} - 1 \right\}} = 1 - \frac{1}{15}e^2 \dots$$

Combined with equation (3) this shows that the deformation produced by the field leads to an increase of the steady velocity under given forces, whose ratio to the whole velocity v is

$$(5) \quad \frac{\Delta v}{v} = \frac{3}{80\pi} \left(\frac{\epsilon - 1}{\epsilon + 2} \right)^2 \frac{aF^2}{T}.$$

¹ Lamb, *Hydrodynamics*, ed. 2, p. 555; referring to Oberbeck, *Crelle's Journal*, LXXXI., p. 62, 1876.

In Millikan's measurements the field was less than 9,000 volts per centimeter ($F < 30$), and the radius of the drop seldom over 6×10^{-4} . With $T = 30$ and a high value for ϵ the correction would thus be in the extreme cases of the order of one part in ten thousand.

No account has thus far been taken of the effect produced by the local field due to the charges on the drop itself, which would depend on their location; but it may be conjectured that this effect also would be negligible unless the total charge were large compared with $\sigma^2 F$, which in most cases would imply a high multiple charge. It is perhaps also questionable whether the formula used for the pressure due to surface tension is accurate on so minute a scale.

UNIVERSITY OF CHICAGO.

June, 1912.

THE MECHANICAL FORCES BETWEEN MOVING ELECTRIC CHARGES.

BY F. R. SHARPE.

IN the April number of the *PHYSICAL REVIEW* Dr. Crehore has given an electrical explanation of the force of gravitation between two atoms, each built up of a positive charge around which an equal negative charge is rotating. The calculations depend on the expressions for the mechanical forces between moving charges which Sir J. J. Thomson obtained in his pioneer paper (*Phil. Mag.*, 1881). The object of this note is to point out that modern theory leads to a force which varies approximately as the inverse cube of the distance and cannot therefore be gravitation. It is also proved that Thomson's expressions can be derived from the modern theory by supposing the velocity of transmission c to increase indefinitely.

The mechanical force exerted at time t on a charge e having velocity $\mathbf{q}$ by a charge e' having velocity $\mathbf{q}'$ and at a distance r at time $t - (r/c)$ is

$$(1) \quad -e \left(\nabla \varphi + \mathbf{q} \times \nabla \times \boldsymbol{\alpha} - \frac{\partial \boldsymbol{\alpha}}{\partial t} \right),$$

where φ is the scalar potential

$$\frac{e'}{\kappa r \left(1 - \frac{\mathbf{q}' \cdot \mathbf{r}}{cr} \right)}$$

and $\boldsymbol{\alpha}$ is the vector potential

$$\frac{\mu e' \mathbf{q}'}{r \left(1 - \frac{\mathbf{q}' \cdot \mathbf{r}}{cr} \right)}.$$

See Abraham's *Elektrizitat*, Vol. II., pp. 18 and 36. The mechanical force found from (1) is given on p. 97. If we allow c to increase indefinitely

$$\varphi \doteq \frac{e'}{\kappa r};$$

$$\boldsymbol{\alpha} \doteq \frac{\mu e' \mathbf{q}'}{r};$$

and the mechanical force (1) approaches the value

$$(2) \quad \frac{ee'r}{\kappa r^3} - \mu ee' \left(\frac{\dot{q}'}{r} + \frac{q'(q' \cdot r)}{r^3} + \frac{q \times q' \times r}{r^3} \right).$$

Since

$$q \times q' \times r = (q \cdot q')r - q'(q \cdot r),$$

the expression (2) may be written

$$\frac{ee'r}{\kappa r^3} - \mu ee' \left(\frac{(q \cdot q')r}{r^3} + \frac{\dot{q}'}{r} + \frac{\{(q' \cdot r) - (q \cdot r)\}q'}{r^3} \right).$$

The first term is the ordinary electrostatic repulsion and the remaining terms are readily identified with Thomson's expressions with the exception of an irrelevant factor $\frac{1}{3}$.

CORNELL UNIVERSITY,

May, 1912.

PROCEEDINGS
OF THE
AMERICAN PHYSICAL SOCIETY.

A METHOD FOR THE PRODUCTION OF A HIGH POTENTIAL FOR ELECTROMETER
WORK.¹

BY A. H. FORMAN.

It is often desirable to obtain a high potential which will be constant enough not to disturb an electrometer, as when one wishes to hold an ionizing chamber at a high potential, and measure the leakage in this chamber with a Dolzolek electrometer. This requires a very steady potential such as might be obtained by a battery of voltaic or storage cells. However, if a very high potential is wanted the size and expense of such a battery is considerable, and it would also require frequent attention to keep it in good working order.

A method which does not offer these objectionable features is to use a battery of condensers and a low source of e.m.f. Trowbridge, Thompson, Lohman and others have used batteries of condensers and caused them to be alternately connected in parallel and series. By this method it is possible to apply a pulsating or intermittent potential to a battery of condensers, giving a ratio of transformation slightly less than the number of condensers employed. The source may be either a steady e.m.f. or a transformer. This scheme may be modified for electrometer work, and the following method has proved satisfactory for this purpose.

The condensers are permanently connected in series, and then *separately* charged in succession from an insulated source of constant e.m.f. This charging in succession is done by means of a rotating contact maker. It consists of a drum with as many pairs of contacts as there are condensers in series. These contact points are staggered on the drum so that but one pair is connected to a condenser at one time, but each pair is always connected to the source of e.m.f. through slip rings on the drum. In this way as the drum is rotated it connects each condenser in succession across the source of e.m.f., and the faster it is rotated the oftener they are charged.

Since the source of e.m.f. is insulated, any point in the series of condensers can be made to have zero potential by grounding it. This will then give any other point in the series a definite absolute potential. And the total difference

¹Abstract of a paper presented at the Washington meeting of the Physical Society, Dec. 27, 1911.

in potential of the series is equal to the number of condensers times the e.m.f. of the source. Also, since each and every condenser is charged for every revolution of the drum, if the speed of rotation is enough, the condensers are kept fully charged.

The advantage of such a method over the use of batteries is evident, both as to cost and as to convenience. Since the potential difference is equal to the number of condensers times the e.m.f. of the battery used, it does not require many condensers nor a source of high e.m.f. The small battery can be kept in condition with little trouble, or if dry cells are used they will require little or no attention, and as the condensers do not deteriorate, they will require no attention at all. One can use ordinary telephone condensers which are cheap and have a capacity of 2 microfarads.

This scheme not only gives a steady potential but it affords a means of calibrating static high potential instruments. The ratio of transformation in this case being equal to the number of condensers, employed, since they are always connected in series.

ITHACA, N. Y.

THE IONIZATION FROM HEATED SALTS.¹

BY CHARLES SHEARD.

INVESTIGATIONS have been carried out on the ionization produced by heating cadmium iodide at temperatures ranging from 345° C. to 480° C. and at pressures of 9–0.4 cm. In the earlier experiments a "double tube" form of apparatus was used. This consisted of two hard-glass tubes, each 30 cm. in length and 2.5 cm. diameter, connected by a short glass cross-connection. The outer electrodes were of platinum gauze fitting closely against the inner glass surfaces and were placed symmetrically with respect to the glass connection joining the tubes. The inner electrodes were of platinum wire. The ionization effects obtained by the use of the double tube apparatus showed that

1. Applying voltages which gave saturation currents in the tube containing the salt, ionization occurred in the second tube. The positive ionization in the unsalted tube was in general comparable in value to that obtained in the tube carrying the salt and in some cases exceeded it. Hence vaporization of the salt without ionization may occur; the vaporized material may be subsequently ionized.

2. The greatest positive emission was obtained initially when the temperature was below the melting point of the cadmium iodide (402° C.); when the temperature was above the melting point there was a gradual rise in current values to a maximum (after 15–25 minutes heating) followed by decay with time.

3. At a constant pressure of 1.3 cm. the maximum positive emission increased in the ratio of 20 : 1 in passing from a temperature of 385° C. to a temperature of 415–420° C.

¹ Abstract of a paper presented before the American Physical Society, New York meeting, March 2, 1912.

4. Several maxima and minima were found in the positive ionization in the unsalted tube.

5. The positive and negative currents in the tube containing the cadmium iodide at a temperature of 355°C. and a pressure of 1 cm. were nearly equal, decaying in each case from an initial value of $75\text{--}100 \times 10^{-10}$ ampere to a steady value of 4.7×10^{-10} ampere in half an hour. In the unsalted tube the negative current decayed from an initial maximum value of $17,745 \times 10^{-10}$ ampere; the positive current fell off with time from an initial maximum value of $2,350 \times 10^{-10}$ ampere.

The existence of a current when the salt-covered plate (usually of platinum) is *negatively* charged has been taken to indicate either the emission of negative ions or volume ionization of the vapors given off by the salt. However, it might equally well arise from the vapors or the sublimed salt at the opposite (central) electrode, resulting in the emission of positive ions which would saturate to the outer negatively charged electrode. Furthermore, chemical reactions between the vapors and the heated electrodes might vitiate the true ionization effects from the salt.

A form of apparatus, in which these points of difficulty were obviated, was constructed and used in some further experimentation. It consisted of a glass tube 40 cm. long, $4\frac{1}{2}$ cm. in diameter, carrying an *air-cooled brass electrode* which passed through the center of a ground glass stopper fitting into the top of the main tube. The second electrode consisted of the glass wall of the tube

TABLE I.

Pressure = 1.8 cm.

Temperature = $470\text{--}475^{\circ}\text{C.}$ Voltage at tube = ± 200 volts.1 div. = 10^{-10} ampere.

Time (Min.).	Negative Current.	Positive Current.	Time (Min.).	Negative Current.	Positive Current.
2	152		40		2.7
3	520		52	3.0	
4	1638		55		3.03
5		0.002	65	2.1	
7	728		68		3.8
8	198		70	1.97	
10	118		73		5.05
12		0.10	83		6.1
13	57		90		5.7
15		1.78	105		3.63
21	26.8		110	1.30	
23		1.90			
29	9.9				
34	8.3				

wrapped *externally* with metal foil, the glass being a good conductor above 300°C. The experiments using this form of apparatus at temperatures ranging from $445\text{--}480^{\circ}\text{C.}$ and at pressures of 3–0.8 cm. showed that

6. There was an emission of *negative* as well as of positive ions. The negative

emission decayed rapidly with time from an initial maximum value. The positive emission increased slowly with continued heating, attained a maximum value after 60–80 minutes and then decayed with time. The ratio of maximum negative to maximum positive emission reached a value as high as 325. The large negative emissions obtained during the first ten or fifteen minutes of heating under potential indicate an expulsion of negative ions, presumably of iodine,¹ from the heated salt.

Some readings from a representative set of data are given in Table I.

7. The distillates were found to be efficient sources of positive and negative ionizations. Definite cycles of maximum negative and positive emissions were shown in the repeated heatings of the distillates. Some results are briefly indicated in Table II.

TABLE II.

Pressure = 0.8 cm. Temperature = 445–465° C. 1 div. = 10^{-10} ampere.

Substance.	Maximum Negative Current.	Maximum Positive Current.
Original CdI ₂	800.0	10.5
1st distillate.....	0.9	30.4
2d distillate.....	92.0	5.3
3d distillate.....	0.9	16.6
4th distillate.....	52.3	4.2

8. The variations in the values of the ionizations produced in the several reheatings of the distillates cannot be attributed to the presence of water vapor. It was found that the addition of water to a sample of the original salt increased the positive but decreased the negative emission.

PALMER PHYSICAL LABORATORY,
PRINCETON UNIVERSITY,
February 20, 1912.

THE EXPANSION OF WATER BELOW ZERO DEGREE CENTIGRADE.²

BY JOHN FRED MOHLER.

THE anomalous behavior of water when it is cooled to near the freezing point has long been known. Calling the volume of water unity at the temperature of its greatest density, its volume at zero, when it freezes is 1.000132 and at $8\frac{1}{8}$ degrees above zero it is the same, while at 4 degrees it is 1, and at 10 degrees it is 1.000273, the curve of volume and temperature showing its minimum at 4 degrees or a little less than 4 degrees, and being steeper on the colder side of the minimum.

It is also known that water in a fine capillary tube will not freeze as readily as water in an open vessel. It occurred to me that I might use this non-freezing property of a capillary tube to get the expansion of water below zero. Several

¹ See abstract of paper on "The Ions Emitted by Hot Salts," by O. W. Richardson, New York meeting, March, 1912.

² Abstract of a paper presented at the Washington meeting of the Physical Society, Dec. 27–30, 1911.

capillary tubes were drawn two and a half meters long with a diameter ranging from .188 mm. to .38 mm. These tubes were folded over several times, then filled with distilled water that had just been boiled, and then sealed at the larger end. This capillary tube was then fastened to a glass tube silvered on the inside and having etched on it a millimeter scale. This scale with the capillary tube attached was then placed in a large vessel filled with cold brine. The position of the column of water in the tube was read with a low power microscope and the brine could be thoroughly agitated so that the temperature was constant throughout the vessel. The temperatures were taken with a thermometer reading to 1/10 degree. Four series of measurements were made with one tube and one with another. The series giving the lowest temperature reached, -13.55 , is given below. The scale readings are mm.; the temperature readings are corrected.

Scale Reading.	Temperature.	Scale Reading.	Temperature.
32.8	+21.8°	33.	-6.60
39.	+ 3.95	34.1	-5.75
39.2	+ 6.05	34.8	-5.1
29.5	- 9.85	35.5	-4.15
30.	- 9.15	36.	-3.45
24.5	-13.55	36.5	-2.9
24.2	-13.55	37.1	-1.85
25.	-13.05	37.7	- .95
26.8	-11.7	38.1	- .15
29.	-10.25	38.5	+ .95
31.	- 8.95	38.9	+2.75
32.1	- 7.85		

A diagram of this set of observations is also given. On the diagram temperatures are measured along the horizontal axis and scale divisions along the vertical axis. The curve *AOB* is the observed curve giving the relative expansion of

T. mp.	Vol.	Temp.
+ 4	1.	+ 4
0	1.000132	8.12
- 1	1.000227	9.4
- 2	1.000327	10.6
- 3	1.000447	11.7
- 4	1.000620	13.15
- 5	1.000825	14.65
- 6	1.001051	16.1
- 7	1.001282	17.45
- 8	1.001501	18.65
- 9	1.001754	19.9
-10	1.002069	21.35
-11	1.002369	22.7
-12	1.002715	24.1
-13	1.003078	25.5

water in glass. Since the volume of water at zero and at eight and one eighth degrees is the same, the line COD drawn through the 0 and $8\frac{1}{8}$ points of the curve will then represent the expansion of glass with negative sign. By adding the difference between the line COD and the horizontal line EOF to the curve AOB we get the curve $HO K$, which is the curve of expansion of water. It will be noticed that it is considerably steeper on the colder side of

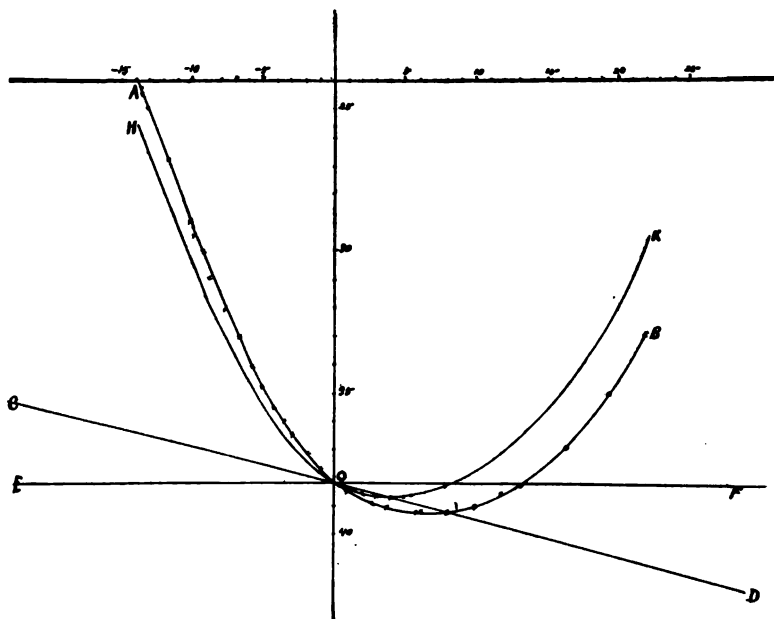


Fig. 1.

the minimum volume. A table of comparative volumes follows. The values for volumes above zero degree are taken from the Smithsonian Physical Tables, fifth revised edition. A similar table is likewise given of the results found by Desprez, Pierre and Weidner, published in Winkelman's *Handbuch der Physik*, Vol. II., Second Part, p. 93.

Mean Value of Volume Given by Desprez, Pierre and Weidner, the Volume at 4 Being 1.

Temperature.	Volume of Water.	Temperature.	Volume of Water.
4	1.	- 5	1.000702
0	1.000125	- 6	1.000890
-1	1.000212	- 7	1.001096
-2	1.000309	- 8	1.001328
-3	1.000422	- 9	1.001586
-4	1.000556	-10	1.001855

DICKINSON COLLEGE,
February 22, 1912.

THE PHYSICAL REVIEW.

IONIZATION AND PHOTO-ELECTRIC PROPERTIES OF VAPORS OF ALKALI METALS.

BY S. HERBERT ANDERSON.

INTRODUCTION.

IT is a well-recognized fact that the photo-electric effect may give some definite information of the nature of radiation and of the distribution of energy in light waves. The importance of this phenomenon makes it very desirable to have reliable quantitative data of the relation between the wave-length of the incident light illuminating a metal and the positive potential acquired by the metal under such conditions. So far the results of different observers show great discrepancies. E. Ladenburg,¹ working in the region λ 2,700 to λ 2,000, obtained data, which, according to his interpretations, show that the relation is a linear one and a verification of Plank's theory of radiation which may be expressed by

$$P = \frac{k_1}{e} n,$$

where P is the potential, e the elementary electrical charge, k_1 a constant and n the frequency of the incident light. Hull,² working in the region λ 1,710 to λ 1,230 obtained data which he interprets in the same way, while Kunz,³ working over a range of λ 5,000 to λ 2,000 obtained data which verify his theory of radiation,⁴ which may be expressed by

$$P = \frac{k_2}{e} n^2.$$

¹ Phys. Zeits., VIII., p. 590, 1907.

² Am. Jr. of Sc., XXVIII., p. 251, 1909.

³ Phys. Rev., XXXIII., p. 208, 1911.

⁴ Phys. Rev., XXIX., p. 212, 1909.

That is, the potential which the illuminated metal acquires varies as the *square* of the frequency and *not* simply as the frequency. More recently Wright¹ has obtained curves for the relation between the wave-length and positive potential which differ both from the results of Ladenburg and Hull and of Kunz, and might indicate that the phenomenon is of a resonance nature which would be in accord with neither Plank's nor Kunz's theory.

Furthermore, the work of Millikan² and Wright (*l. c.*) shows that the magnitude of the photo-electric effect is largely determined by surface conditions. Gehrts³ has shown that the reflection of electrons and secondary electrons may produce effects which amount to 40 or 50 per cent. of the original effects; and consequently the methods of investigation must be such as to avoid these. Hughes⁴ has shown that metal surfaces procured by distillation of the metals *in vacuo* give a larger photo-electric potential than is obtained from polished surfaces which have not been treated to a glow discharge, and is much more stable and constant. The work of all these observers shows that the photo-electric effect is far from a simple one. The exact law cannot be known until all the factors entering in are known and understood.

In the physical laboratory of the University of Illinois considerable work has been done upon the photo-electric effect of alkali metals. The advantages of work with these metals are: (1) a clean surface can easily be obtained by distillation *in vacuo*, and (2) these metals give larger photo-electric currents than any other metals. In investigations carried on by Professor Kunz, Dr. J. G. Kemp and the author, it has been observed that there is present in tubes prepared as photo-electric cells, containing alkali metals, a conduction due to something else than the electron current arising from the photo-electric action. It was thought that possibly this was due to ionization of the vapor of the alkali metals. Therefore the following investigation was undertaken to determine (1) whether or not there was a spontaneous ionization of the vapors of alkali metals and (2) to determine the magnitude of the currents of conduction for different temperatures.

Füchtbauer⁵ has made some investigation of the spontaneous ionization of sodium vapor at temperatures ranging from 190° C. to 330° C. In a later paper⁶ he gives some results obtained with cæsium vapor, but in

¹ *Phys. Rev.*, XXXIII., p. 43, 1911.

² *Phys. Rev.*, XXXIV., p. 68, 1912.

³ *Ann. der Phys.*, 36, p. 995, 1911.

⁴ *Proc. Camb. Phil. Soc.*, XVI., p. 167.

⁵ *Phys. Zeits.*, 10, p. 374, 1909.

⁶ *Phys. Zeits.*, 12, p. 225, 1911.

an atmosphere of helium of 196 mm. pressure, which would render the results quite different from those obtained with cæsium vapor alone. The temperatures he used for cæsium were from 150° C. to 210° C. In the former paper he gives a curve of conduction in sodium vapor which has very little of the characteristic features of an ionization curve. In the latter paper curves are obtained which do approximate to ionization curves.

Fredenhagen¹ in an investigation on the electrons given off from sodium and potassium made some observations of the conductivity of the vapors of these metals. He heated the potassium vapor up to 420° and the sodium up to 500° and found no currents that were measurable with a galvanometer that gave a deflection of one millimeter for a current of 3.7×10^{-10} amperes. The tube used for this part of his work was of the same type used by the author (Fig. 1, A).

The investigations of Führtbauer and Fredenhagen do not throw much light on the photo-electric phenomenon. One cannot tell whether the conductivity they obtained is due to the vapor alone or to particles given off from the metal which was present in the tube containing the electrodes. And it is especially desirable to know whether or not there is a spontaneous ionization at ordinary temperatures, 20° to 25° C.

EXPERIMENTAL METHOD.

Two types of tubes were used in this investigation. In the first, Fig. 1, A, the two electrodes *a* and *b* were of the same material, viz., nickel, and of the same dimensions. In the second, Fig. 2, one electrode, *b*, consisted of the alkali metal, potassium or cæsium; the other, *a*, was a disk of nickel.

In using the tube of the first type potassium was placed in bulb *c* (Fig. 1), and a series of measurements taken when the field was from *a* to *b*, and also for the field from *b* to *a*, for the temperatures 25°, 50°, and 100° with the tube in darkness. Then the potassium was poured into the main tube and lodged between the electrodes, but not in contact with either of them, and a series of measurements taken for 25°, 50°, 100° and 150° with the tube in darkness and also with a beam of light falling on the potassium, but not on the electrodes.

With the tube of the second type (Fig. 2) a series was taken with potassium for one electrode for 25°, 50°, 100° and 150° both "in darkness" and with the "potassium illuminated." The fourth series of measurements were taken with the tube of the second type and with cæsium for one electrode for temperatures of 25°, 40°, 70° and 100°.

¹ *Phys. Zeits.*, 12, p. 398, 1911.

EXPERIMENTAL DETAILS.

Construction and Preparation of the Tubes.—Tube No. 1 (Fig. 1, A), consisted of a glass tube 16 cm. long, 3 cm. in diameter at the middle. The two electrodes were nickel disks 19 mm. in diameter and 2 mm. thick. The distance between the electrodes was 2 cm. Nickel was used for the electrodes because it probably contains less gas, especially oxygen, at ordinary temperatures and pressures than platinum or silver and does not tarnish very readily.

In tube No. 2, Fig. 2, the bulb was 4 cm. in diameter. The upper electrode was a nickel disk 19 mm. in diameter and 2 mm. thick. The lower electrode was potassium for one set of experiments and caesium

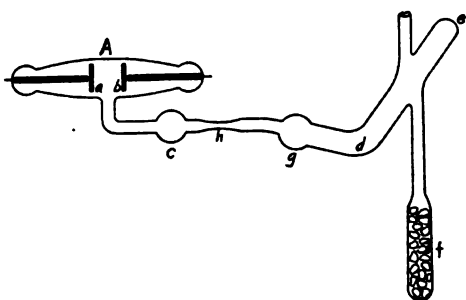


Fig. 1.

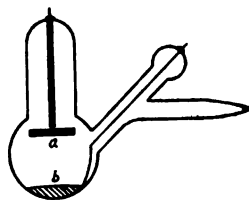


Fig. 2.

for another. The area of the potassium was about the same as that of the nickel electrode; and of the caesium about 1 cm^2 . The distance between the electrodes was 2 cm.

For the preparation of potassium the entire tube as shown in Fig. 1 was used. A piece of potassium was introduced into the tube through an opening at *e* and placed at *d*. The opening at *e* was then closed, and exhaustion by a Gaede pump begun. During the evacuation the tube *A*, containing the electrodes, was enclosed by a heating coil and kept at a temperature of 200°C . for about three hours, in order to remove as much as possible the gases occluded in the walls of the tube and in the electrodes, so that on subsequent heating gas would not be given off. The charcoal bulb *f* was heated to 300° or 400° during the exhaustion. When it was possible to pump the system down to a soft Röntgen ray vacuum with *A* and *f* still heated, then the heating coil was removed from *A* and the gas flame from under *f* and the exhaustion carried to a hard Röntgen ray vacuum. The vacuum was tested by a discharge tube not shown in the diagram. The potassium was melted and the oily vapors from the crust pumped out during the melting. Potassium was then distilled into the bulb *g* in the following manner:

On carefully heating the potassium at *d* vapor went over into *g* and metallic potassium deposited on the walls of the bulb. The tube was then allowed to cool down and the process repeated. In this way sufficient potassium could be collected in *g*. The surface of the potassium obtained in this manner is as brilliant as polished silver. The whole system shown in Fig. 1 was sealed off from the pump, the potassium poured into bulb *c* and the charcoal bulb placed in liquid air. After waiting 20 or 30 minutes for the charcoal to absorb the remaining gas the tube *A* was sealed off from the rest of the system at *h*. By this method of preparing a tube as good a vacuum as possible is obtained.

The method of preparing tube No. 2, Fig. 2, was the same except that the potassium was poured into the tube and lodged at *b* before it was sealed off from the rest of the system.

A similar method was used in preparing tube No. 2 with cæsium for one electrode, but with the following modification: the tube (Fig. 1) was closed with a rubber stopper at *e* and exhausted. Then the whole system including the pump was filled with nitrogen at atmospheric pressure. A small bulb in which the metallic cæsium was contained was filled with nitrogen, the cæsium melted and poured into tube at *e*, which was then closed and exhaustion begun. From this point on the method of preparing the tube was the same.

Arrangement of Apparatus for Measuring Conductivity at Various Temperatures.—The method used for measuring the conductivity of the alkali vapor is shown diagrammatically in Fig. 3. The tube, either No. 1 or No. 2, was placed in the heating coil *H*, which was enclosed in a sheet-iron box *C*. This could be made light tight and the experiment was carried on in a dark room.

The copper core of the coil and the sheet-iron box were both grounded. A wire from the electrode *a* passed through an insulating plug and was joined to a movable wire *W*, which made connection with a water rheostat *R*. To the water rheostat were connected the terminals of a storage battery *B*₁ of 1,800 volts and one

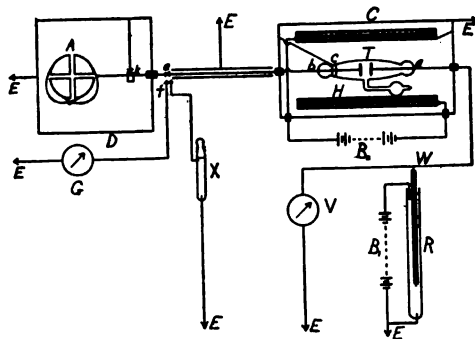


Fig. 3.

terminal of the battery was grounded. From electrode *b* a wire passed through an insulating plug and was connected to one pair of quadrants of a Dolezalek electrometer. The other pair of quadrants was grounded.

k is a plunger key by which the electrode b and the pair of quadrants to which it is connected may be grounded. c is a collar of metal foil placed around the tube near the end and connected to earth so that conduction over the outside of the tube may be avoided. The electrometer and connecting wires were enclosed in a sheet-iron box D , which was grounded.

The wire connecting electrode b to the electrometer was enclosed in a metal tube between C and D and this was also grounded. The scheme described above was used when the potential of electrode b was to be measured, and for measuring currents of the order 10^{-13} to 10^{-11} amperes. When currents of the order 10^{-11} to 10^{-9} were to be measured one terminal of a high resistance X was connected to the wire running from b to A at e . The other terminal of X was grounded. When k was opened a current flowed from b through X to the earth and the electrometer deflection gave the fall of potential across this resistance. And from these two quantities the current can be computed. When currents greater than 10^{-9} were to be measured the wire from b was disconnected from the electrometer at e and connected to a wire f from a galvanometer. The other terminal of the galvanometer was grounded.

The potential applied at a was measured by one of three electrostatic voltmeters V of the Kelvin and White type. Three voltmeters were necessary to cover the range required by the investigation; the first read from 0 to 90 volts, the second from 100 to 600, the third from 500 to 1,500.

When it was desired to pass a beam of light into the tube a slit in the box C was opened which was in line with the slit through the heating coil.

Heating Coil.—The tube was kept at a temperature higher than room temperature by means of a heating coil or electric furnace. The core of the furnace consisted of a copper cylinder 10 cm. in diameter and 25 cm. long. At the middle of the cylinder two slits were cut opposite to each other, 1.5 cm. by 0.5 cm., through which a beam of light could be directed into the tube placed inside the coil. The winding of the coil consisted of two layers of wire of the same number of turns so connected that the magnetic fields of the two layers were opposed. With this arrangement the magnetic force within the cylinder was found to be so slight as not to effect appreciably the movement of the ions in the tube.

Galvanometer.—The galvanometer used was a D'Arsonval instrument manufactured by W. G. Pye, Cambridge. The resistance was 122 ohms, and its period 9 seconds. The deflections were read by a telescope and scale at a distance of 2.5 meters. One millimeter deflection indicated a current of 0.837×10^{-9} amperes.

High Resistance.—The high resistance used was a liquid resistance, consisting of meta-xylyl with a few drops of absolute alcohol. The

resistance could easily be changed by varying the proportion of meta-xylol and alcohol. The container consisted of a glass tube 0.9 cm. in diameter and 8.0 cm. long, with two platinum electrodes sealed through the glass 3.5 cm. apart. The tube was closed by a ground glass stopper. The resistance was observed to increase with the time probably due to evaporation of alcohol. But the resistance was constant for a period of a few hours required for a set of readings. The resistance used varied from 4.72×10^9 ohms to 14.3×10^{10} ohms.

EXPERIMENTAL RESULTS.

Tube No. 1. Potassium in Bulb c.—The first series of measurements which will be presented were taken with tube No. 1 (Fig. 1, A), with the potassium in bulb *c* and with the tube in the dark. The method of taking the readings was as follows: The potential was applied at electrode *a* with the key *k* closed, so that the charge which would arise from electrostatic induction in electrode *b* and connecting wire would be removed through connection to earth at *k*. Potentials as high as 1,690 volts were applied. At 25°, 50° and 75° C. *nothing of the character of a regular conductivity was observed. But a phenomenon which has not been reported before was observed.* On opening the key *k*, after a *positive* potential was applied at *a* the electrometer gave a positive deflection which very soon came to a maximum. *Then there was a negative deflection which was nearly of a constant rate.* A test was made to determine whether this was due to a charge or a continuous current. With +100 volts applied at *a* this movement of the electrometer needle toward the negative continued for three hours and fifty minutes, and the rate of deflection was nearly constant for the first fifty minutes. After the maximum negative deflection had been reached the electrometer needle drifted toward the zero at the rate of 0.5 mm. per minute, which was about the rate of the natural leak. So we would conclude that the effect is of the nature of a charge which at the temperature of 25° builds up very slowly and at first the rate of increase is a linear function of the time.

When a negative potential was applied the results were similar except that the directions of deflections were reversed. It was also observed that if a higher potential was applied after observations had been made for a given potential the "negative" current was not as large as was the case when a period of rest was given the tube with both electrodes grounded. Consequently it was found necessary in taking a set of readings to leave the tube earthed by both electrodes for 10 minutes between the measurement of the current for a given potential and the measurement for the next higher potential. Furthermore it was noticed

that if the tube was exposed to light while a potential was applied at *a*, there was a larger positive deflection and after the light was turned off the "negative" current was larger than before. After sufficient time in the dark the "negative" effect decreased to a constant value.

TABLE I.

"Positive" Charge and "Negative" Current at 25°.			"Positive" Charge and "Negative" Current at 50°.		
Potential in Volts.	"Positive" Charge.	"Negative" Current.	Potential in Volts.	"Positive" Charge.	"Negative" Current.
+ 100	+ 4.3	-0.4	+ 100	+ 2.5	- 6.2
300	12.0	1.2	300	6.5	16.2
500	19.8	2.0	500	14.0	24.8
700	25.7	2.8	700	18.0	36.0
900	35.5	3.5	900	20.5	47.0
1,100	44.4	4.5	1,100	25.0	58.0
1,300	—	—	1,300	28.0	72.0
1,490	off scale	5.5	1,490	32.0	110.7
1,690	" "	6.2	1,690	off scale	111.0
- 100	- 3.8	+0.47	- 100	- 2.5	+ 5.6
300	12.6	1.2	300	7.5	16.0
500	22.8	1.9	500	12.0	26.6
700	29.9	2.55	700	16.5	37.0
900	39.0	3.0	900	21.0	47.4
1,100	49.9	3.6	1,100	25.0	58.0
1,300	off scale	4.3	1,300	31.0	66.0
1,490	" "	5.0	1,490	31.5	84.0
1,690	" "	5.2	1,690	39.0	97.0

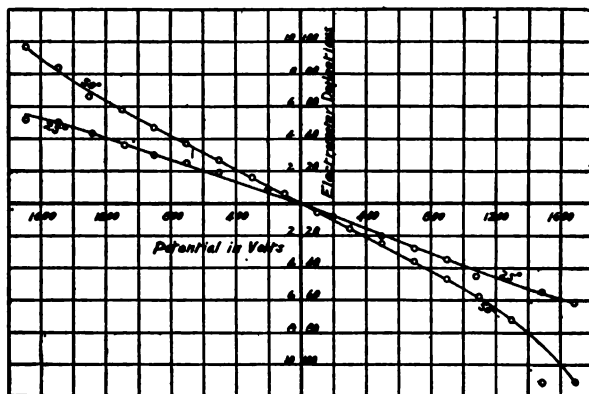


Fig. 4.

In Table I. there are given data which show the relation between the potential applied and the "positive" charge, that is, the first electrometer deflection after opening the key, which is in the same direction as the

field; and also the relation between the potential applied and the "negative" effect, which is a current appearing after the "positive" charge reaches a maximum and is in a direction *opposite* to the field. The "negative" current was measured by the rate of deflection of the electrometer. The "negative" current for 25° and 50° is shown by the curves of Fig. 4. For the curve for 50° the ordinates have 10 times the value that they have for the curve for 25° . *The remarkable feature of this phenomenon is that even with a potential difference of 1,700 volts there is a current in the opposite direction to what we would expect from our present knowledge of conductivity.*

When the tube was heated up to 75° the "positive" charge did not appear, but immediately on opening the key there was a "negative" deflection. The initial negative current was much larger than at lower temperatures, but the effect showed more the character of a charge in that it came to a maximum much sooner.

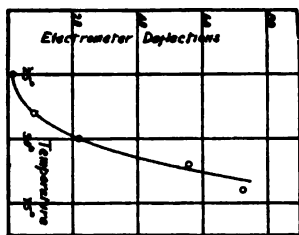


Fig. 5.

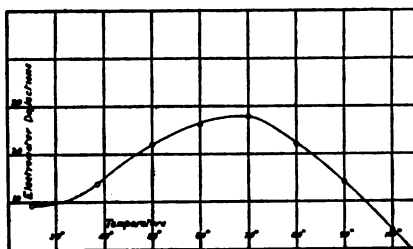


Fig. 6.

The variation of the initial "negative" current and of the charge with the temperature is shown by the curves of Figs. 5 and 6 respectively. In taking the readings for these curves a constant potential of + 400 volts was maintained. It was impossible to measure the currents for temperatures higher than 75° by this method. The "charge" curve, Fig. 6, shows that the effect reaches a maximum between 65° and 70° . If the curve is extended until it cuts the axis of abscissæ, it would appear that the effect disappears at about 104° .

At 100° when the potential was gradually increased up to + 700 volts a reversal of the electrometer deflection suddenly occurred and there was a *very large* positive deflection accompanied by a lighting up of the tube. There is then at this temperature and pressure a conductivity of the usual character through the vapor.

In order to determine whether or not this "negative" effect was due to anything else besides the potassium vapor, a new tube was made of the same form and dimensions in which no potassium was present. The

new tube was made with new electrodes and from new glass tubing thoroughly cleaned. The cocoanut charcoal used for exhaustion was freshly burned. The tube was pumped out with a new Gaede pump which had never been used for the exhaustion of tubes where alkali metals were present. The tube was pumped out to a hard Röntgen ray vacuum, the charcoal bulb immersed in liquid air for about 20 minutes and then the tube was sealed off. This tube showed the same "negative" effect as the one with potassium vapor. When first tried immediately after sealing off, it gave no positive deflection when a positive potential was applied, but a negative deflection the rate of which was about five times as great as that observed in the former case. After leaving in the dark and grounded by both electrodes for 10 hours there was then observed, when a positive potential was applied and the key opened, first a positive charge which was followed by a negative current which was smaller than the one observed when the tube was first tried. When a 16-candle-power incandescent lamp shone on the tube, the negative current increased and the positive charge entirely disappeared. After leaving in the dark for a few hours the positive charge reappeared and the negative current decreased. Hence it seems that this phenomenon (1) arises from the glass tube or the metal electrodes, (2) is affected by light, (3) is diminished by the presence of potassium vapor. Four tubes with similar electrodes of nickel and containing potassium vapor, one tube with nickel electrodes and no alkali vapor, and one tube with aluminum electrodes and no alkali vapor, *all showed the same phenomenon.*

No attempt is made at this time to explain this effect, but the observations made are presented and must be taken account of in measurements of small conductivities through gases.

Tube No. 1. Potassium between the Electrodes.—As the vapor of potassium shows no spontaneous ionization for the range of potentials and temperatures used, the next problem that presents itself is whether or not the potassium gives off particles which are carriers of electricity. For this investigation the potassium in bulb *c* was melted and poured into the main tube and lodged between the electrodes, but not in contact with either of them. Measurements were made of the currents for temperatures 25°, 50°, 100° and 150° with the tube in darkness and with the potassium illuminated by a beam of light. The source of light was a 16-candle-power incandescent carbon filament lamp, supplied from a storage battery of 120 volts. The beam of light did not fall on the nickel electrodes, though of course some reflected light did reach them. The results obtained are shown by the curves of Fig. 7. The curves with solid lines represent the currents with the "potassium in darkness," those

with broken lines represent the curves with the "potassium illuminated."

Attention is called to four particulars of these curves: (1) the "negative" effect persists for 25° and 50° in darkness and is the same order of magnitude as occurred when the potassium was in bulb *c*; this disappears for 100° in darkness and is not apparent for any of the temperatures tried

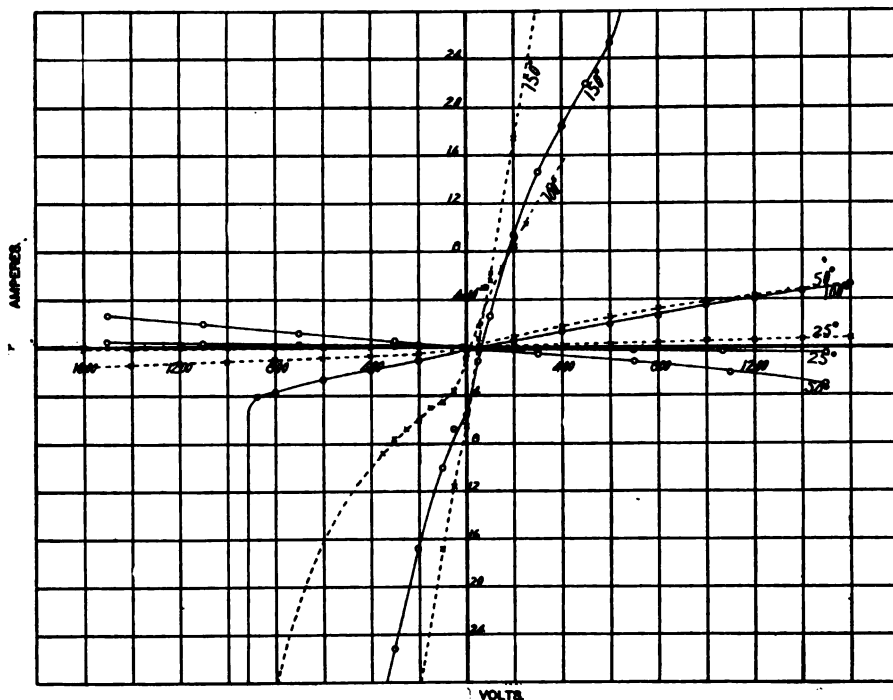


Fig. 7.

when the potassium is illuminated; (2) the linear character of the curve for 100° in darkness and the abrupt bend in the negative branch at 875 volts; (3) for 25°, 50° and 100°, "potassium illuminated," the positive branches of the curves show larger currents than the negative; (4) for 100° and 150° there is a negative current when no potential is applied. These peculiarities can be better explained later after the presentation of the phenomena of tube No. 2.

Tube No. 2. Potassium.—This tube gives more nearly than tube No. 1 the conditions that are realized in a photo-electric cell prepared for determining the maximum positive potentials assumed by a metal under the action of light. The conductivity was measured with the tube "in darkness" and with the "potassium illuminated" at the temperatures of 25°, 50°, 100° and 150°. The results are shown by the curves of Fig. 8.

The solid lines represent the currents with the "potassium in darkness," the broken lines the currents with the "potassium illuminated." For both of the curves for 150° the ordinates have 10 times the value marked on the figure, that is, the unit is 10^{-9} instead of 10^{-10} , which applies to the other curves.

One of the most interesting points brought out by this series of measurements is the existence of a negative current when no electric field is

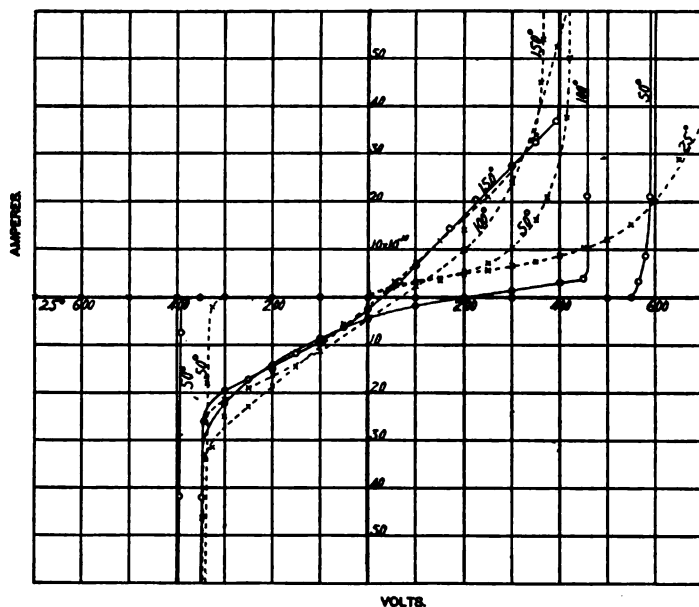


Fig. 8.

applied; that is, there is a positive current from b to a , Fig. 2, from the potassium electrode to the nickel electrode. This phenomenon was observed and has been investigated by J. W. Woodrow¹ in this laboratory. This effect is apparent at 25° in darkness and at 50° , 100° and 150° with the potassium illuminated as well as in darkness. Woodrow found this effect characteristic of the alkali metals. This effect for potassium and caesium is shown by the curves of Figs. 9 and 10. The curves of Fig. 9 show how the negative charge on the potassium or caesium increases with the time. In order to obtain the readings for these curves the alkali metal electrode, b , was connected to the electrometer and a was earthed. The curves of Fig. 10 show the increase of the current with the temperature. (The ordinates for the potassium curve have a larger value than for the caesium curve.) These curves are introduced here because of

¹ *Phys. Rev.*, XXXV., p. 203, 1912.

their bearing upon the author's results and as a confirmation of Woodrow's observations.

The second point of interest of this tube is that at 25° "in darkness," when a given potential is applied at *a*, the current occurring is not con-

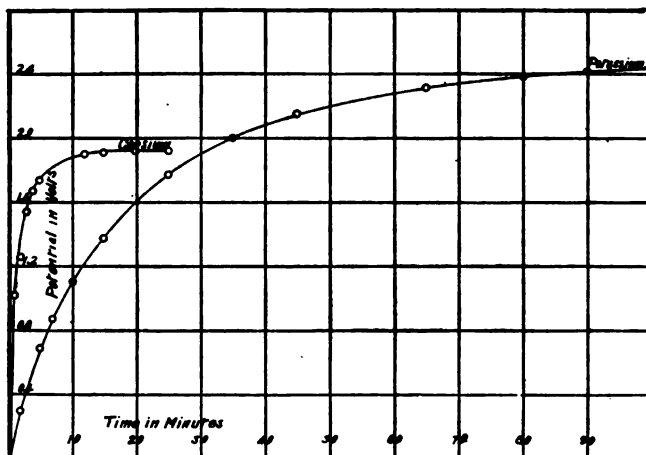


Fig. 9.

stant. This is shown by the curves of Fig. 11. Curve *B* shows how the current varies when + 300 volts is applied, and curve *B'* when - 300 volts is applied. *B'* could not be carried farther as the deflections went

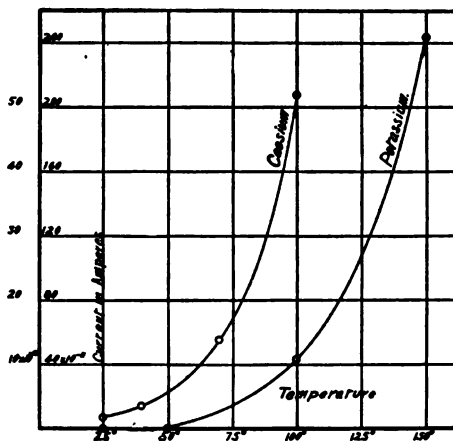


Fig. 10.

off the scale. When no potential is applied, that is, when *a* is grounded and *b* is connected to the electrometer the negative current, mentioned above, decreases as the charge on the electrode *b* builds up. This is shown

by curve *A*. When the ordinates of *A* are subtracted algebraically from those of *B* and *B'* two curves *C* and *C'* respectively are obtained which are almost identical, except that ordinates are of opposite sign. Hence it

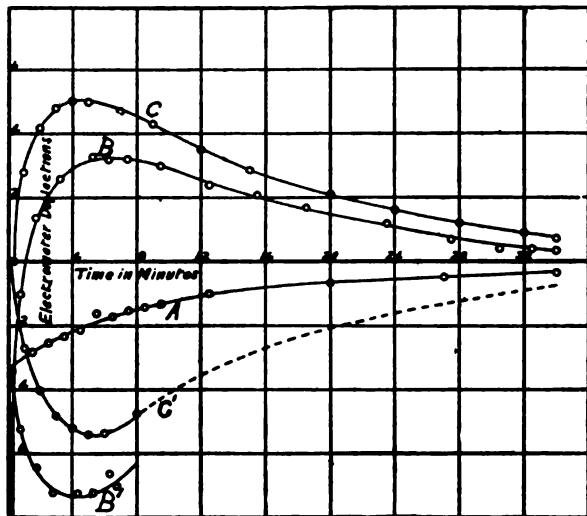


Fig. 11.

would appear that the difference between *B* and *B'* is due to the effect of *A*.

The equation of a curve of the type of *C* and *C'* is of the form

$$i = Ae^{-at} + Be^{-bt} + C.$$

The constants of this equation can be obtained by taking values of *i* for five different values of *t* and thus getting five equations with the five unknowns *A*, *B*, *C*, *a*, and *b*. The actual values of these constants are of no consequence as they will give us no further information of the nature of the phenomenon; so the determination of the constants was carried out only far enough to see that all five constants are real and different from zero. These curves then may be represented by an equation of this form. From this fact we may conclude that there are *three* distinct effects occurring in the tube in addition to the current of curve *A*: (1) a positive current which decreases according to the equation

$$i = Ae^{-at},$$

(2) a negative current decreasing according to

$$i = Be^{-bt},$$

(3) and a constant current given by

$$i = C.$$

TABLE II.

Tube No. 2, Cesium.

Temperature 25°.

In Darkness.		Cesium Illuminated.	
Potential in Volts.	Current in Amperes.	Potential in Volts.	Current in Amperes.
0	$- 1.83 \times 10^{-12}$	0	$+ 0.58 \times 10^{-11}$
+ 200	+ 1.502	+ 100	10.56
- 400	3.34	300	15.4
600	6.44	500	18.0
800	11.27	700	21.5
1,000	16.85	900	24.1
1,200	23.85	1,000	25.4
1,400	33.5	1,640	192.0
1,600	38.3		
1,650	40.2		
- 200	- 3.35	- 200	- 0.963
400	5.47	400	1.26
600	8.52	600	1.705
800	13.6	800	2.295
1,000	20.2	1,000	3.37
1,200	30.1	1,200	4.77
1,400	47.0	1,400	7.03
		1,600	11.5
		1,650	13.2

Temperature 40°.

0	$- 5.94 \times 10^{-12}$	0	$+ 0.222 \times 10^{-11}$
+ 100	2.58	+ 10	14.2
200	1.29	88	23.1
300	1.4	410	83.7
400	+ 6.87	500	543.0
500	16.8		
600	22.6		
625	60.2		
- 100	- 7.76	- 12	- 4.82
200	9.9	57	6.33
300	19.5	100	8.15
400	41.2	200	8.7
600	837.0	300	11.85
		366	25.6
		550	83.7
		600	234.0

TABLE II.—*Continued.*

Temperature 70°.

In Darkness.		Cesium Illuminated.	
Potential in Volts.	Current in Amperes.	Potential in Volts.	Current in Amperes.
0	$- 14.0 \times 10^{-10}$	0	$- 0.036 \times 10^{-10}$
+ 100	1.2	+ 100	+ 1.88
200	+ 15.6	200	2.48
300	78.8	300	3.35
370	232.0	400	83.7
		500	167.5
- 100	- 0.293	- 100	- 0.547
200	0.58	197	2.53
300	1.7	300	14.2
321	2.53	400	25.05
420	217.0	500	201.0

Temperature 100°.

0	$- 0.5 \times 10^{-10}$	0	$- 0.248 \times 10^{-10}$
+ 100	+ 0.1	+ 100	1.43
200	0.9	190	2.68
300	2.64	300	12.55
400	10.05	400	20.05
500	23.4	500	38.2
600	134.0	600	41.8
665	655.0	800	100.5
		1,000	309.0
- 100	- 1.09	- 100	- 1.6
200	2.25	200	4.18
300	3.35	300	8.37
400	10.9	400	16.75
500	25.01	600	67.0
600	45.2	800	159.0
700	75.3	1,000	259.0
825	250.5		

Curves *B* and *B'* are typical of all those taken for various potentials. For negative potentials of 500 volts or more the initial current is positive; this decreases and is followed by a negative current which increases to a maximum and then decreases.

The features of the curves of Fig. 8 to which attention will be called later are: (1) with the "potassium illuminated" all the curves obtained when a positive potential was applied at *a* show the characteristics of ionization curves, though saturation is not clearly marked; (2) the negative branch of the curve for "potassium illuminated" at 25° shows practically no current; (3) the abrupt bend away from the axis of abscissæ

in all the curves obtained when a negative potential was applied; (4) the curves "in darkness" and "potassium illuminated" very nearly coincide for the negative branches of the curves at 100°, and for both the positive and negative branches of the curves at 150°.

Tube No. 2. Cæsium.—The observations in this series were carried out in the same manner as for tube No. 2, potassium, except that the temperatures used were 25°, 40°, 70° and 100°. The results are given in the data of Table II. No curves are shown because they differ in no essentials from those for potassium, Fig. 8. The following features are worth noting: (1) the existence of a negative current when no field is applied, as with the potassium, due to the emission of positive particles from the cæsium; (2) the great increase in the current at 25° and 40° when the cæsium was illuminated over the currents occurring when the cæsium was in darkness.

DISCUSSION OF RESULTS.

The experiment with tube No. 1 with potassium alone between the electrodes, that is, with the solid potassium in bulb *c*, Fig. 1, shows that there is nothing of the character of a spontaneous ionization at a temperature as high as 75°. No conductivity of the usual type was observed when a potential of 1,700 volts was applied. At a temperature of 100° when the potential is gradually increased there is a luminous discharge at 700 volts. But there is no current before this occurs. The abruptness of the discharge indicates that there are no ions present until produced at this instant. At this temperature, 100°, the velocity of agitation is relatively large. Some molecules acquire sufficient velocity so that when a collision with another molecule occurs the electron system of the atom is put in such an unstable condition that the electric force due to the 700 volts difference of potential between the electrodes is sufficient to drag an electron from the atom and ionization occurs. The ions thus produced acquire sufficient velocity under the action of the electric field to produce ionization by collision and the luminous discharge occurs.

The absence of spontaneous ionization is further verified by the fact that when in tube No. 1 the potassium was introduced into the tube so as to lie between the electrodes no conductivity of the usual type occurred until the temperature was raised to 100°.

Woodrow found that the alkali metals in the dark gave off positive particles and that the current arising therefrom increased with the temperature according to the relation

$$i = aTe^{-b/T},$$

where a and b are constants and T the absolute temperature (see curves of Fig. 10). This will be used as a basis for explaining the peculiarities of the curves of Fig. 8, to which attention was called earlier.

The curve for 100° "in darkness" is a straight line up to 1,600 volts. It does not seem reasonable to assume that this is a portion of an ionization curve or saturation would be reached before 1,600 volts. Since positive particles are given off from the liquid potassium at this temperature in some considerable numbers, we may assume that from the surplus of electrons left in the metal some will be given off under the action of the field between the electrodes. When equilibrium is reached the same number of positive and negative particles will be given off and there will be a steady current. The number of particles given off will increase with the field and if the relation is a linear one the curve is explained. Since in all probability the positive ions are atoms of potassium, potassium will be deposited on electrode b . If this is the case the other peculiarities of the curves of Fig. 7 can be explained.

After this curve is taken with the positive potential the curve with the negative potential is taken at the same temperature. Now with the field in this opposite direction to what it was formerly positive particles will be given off from the layer of potassium on electrode b in addition to positive and negative particles given off from the potassium in the tube. The direction of the field will facilitate the natural tendency of the potassium to give off positive particles. This accounts for the fact that in the negative branch of the curve for 100° "in darkness" the currents are a little larger than in the positive branch. Furthermore at 875 volts there is an abrupt bend in the curve showing ionization by collision. At this potential the positive particles acquire sufficient velocity to ionize the vapor. And increase of ionization is very rapid when positive ions begin to produce ionization by collision.

For 150° the order of taking observations was the same as for 100° and this would result in a greater deposit of potassium on electrode b . The negative branch of the curve "in darkness" becomes parallel to the current axis for a lower potential than the positive branch. And this is accounted for by the fact that positive particles are given off from electrode b when the field is in this direction.

Attention has been called to the fact that for 25° , 50° and 100° the positive branches of the curves for "potassium illuminated" give larger values for the currents in the linear portions of the curve than do the negative branches. This can be explained also by assuming that there is some potassium deposited on b . For when the direction of the field is from a to b under the action of reflected light electrons will be given off,

while they will not be given off so readily when the field is from *b* to *a*.

The negative current at 100° and 150° with no field present may also be explained by the deposit of potassium on electrode *b*.

Of course this deposit of potassium on *b* is of no vital importance in the investigation, but the peculiarities of the curves of Fig. 7 are very reasonably explained by this fact.

Attention previously has been called to the fact that some of the curves of Fig. 8, taken with tube No. 2, potassium, show the characteristic features of ionization curves, viz., (1) a straight line where the current obeys Ohm's law, followed by a bending toward the axis of abscissæ, (2) a portion parallel to the axis of abscissæ where saturation occurs, (3) a rapid rise away from the axis of abscissæ where ionization by collision begins. The curves obtained differ from the ionization curve above described in that at no point does the curve become parallel to axis of abscissæ. This is due to the source of ions. In order to have a portion of the curve parallel to the axis of abscissæ, the source of ions must be such that the rate of producing these primary ions is constant. This is the case where the gas is ionized by Röntgen rays or a radioactive substance. In this experiment the source of the ions is the photo-electric action of the potassium, and the rate of production increases with the potential applied. This accounts for the straight line portion of the curve seen in the curves of Fig. 8. The point where the curve begins to bend away from the straight line corresponds to the potential where ionization by collision begins. In the curves taken at 100° and 150° the straight line portion of the curve is not so apparent because at these temperatures ionization by collision begins at a lower potential.

TABLE III.

In Darkness.

Temperature.	Current.				
	0 Volts.	100 Volts.	200 Volts.	300 Volts.	400 Volts.
25°	— 0.0283×10^{-11}	— 0.035	— 0.055	— 0.071	— 0.101
50°	— 0.59	— 0.556	— 0.47	— 0.38	— 0.4
100°	— 44.0	— 22.0	— 1.56	+ 15.0	+30.0
150°	—244.0	+670.0	+1,780.0	+2,760.0	off scale.

Potassium Illuminated.

Temperature.	Current.				
	0 Volts.	100 Volts.	200 Volts.	300 Volts.	400 Volts.
25°	— 0.075×10^{-10}	+ 3.2	+ 5.0	+ 6.5	+ 8.7
50°	— 0.242	+ 2.6	+ 5.0	+ 10.5	+ 28.5
100°	— 4.3	+ 2.3	+ 9.6	+ 23.8	+ 58.5
150°	—21.0	+70.0	+158.0	+250.0	+1,010.0

The curves taken with caesium are similar to those with potassium.

The negative branch of the curve taken at 25° with the "potassium illuminated" shows practically no current of the same magnitude as the other curves. This is what would be expected; for with the field in that direction no electrons would be given off. Any current that occurs must be due to the positive particles. It is reasonable to expect that if a

higher voltage had been applied there would have occurred an abrupt bend as in the curve for 50° . Such a bend occurs where the positive particles acquire sufficient velocity to produce ionization by collision.

The close agreement of the curves for "in darkness" and "potassium illuminated" in the negative branches for 100° and 150° shows that the source of ions must be the positive particles whose expulsion from the potassium is independent of the action of light.

It is rather remarkable that for the curves for 50° , 100° and 150° the abrupt bend occurs at the same potential very nearly, 350 volts. This indicates that

the velocity acquired to produce ionization by collision is due to the electrical field almost entirely and that the initial velocity for all temperatures is small compared to this.

In order to show how the currents obtained with tube No. 2, with electrode *b* of potassium vary with the temperature we may plot temperatures as abscissæ for a given potential. These curves are given in Fig. 12, and the data for them in Table III.

The solid lines represent currents measured with the potassium "in darkness." The portions of the curves represented by dots and dashes have been determined only in a qualitative way. Reference to the table will show that there is some warrant for drawing them as they are. The curves of broken lines represent the currents measured with the "potassium illuminated." The ordinates for the latter have 40 times the value of the former. For all these curves the field is from *a* to *b*, so that the

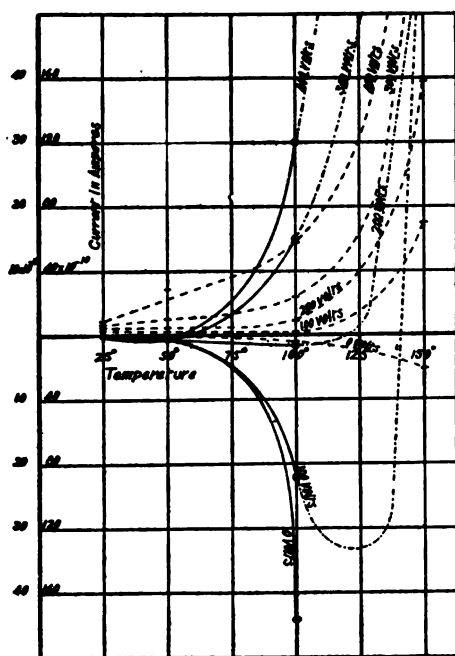


Fig. 12.

positive currents are due to electrons given off from the potassium or to ions produced by them, while the negative currents are due to the positive particles. The curves show that for temperatures up to 50° for the potassium in darkness the emission of positive particles is predominant for all potentials. But for higher temperatures the emission of electrons is predominant for potentials of 300 volts and more. It is quite likely that the critical temperature is the melting point of potassium, 62° .

If the curve for 0 potential, "potassium illuminated," were plotted on the same scale as the curve for 0 potential, "potassium in darkness," it would be seen that the two very nearly coincide, as the data of the table show. This means that the electron current due to the photo-electric effect is very small compared to the emission of positive particles from the potassium at higher temperatures. The greatest relative difference between the two effects is at 25° . At this temperature when the light is applied the current changes from a small negative value to a positive value 25 times greater. But for higher temperatures the resulting current is negative both with and without light and the difference between the two currents is only of a few per cent. However, when a field is applied from a to b then the electron current becomes predominant.

QUANTITATIVE RESULTS.

Ratio of the Number of Electrons Given off to the Number of Atoms Present.

—From the electron current occurring when the potassium is illuminated either with or without an applied potential, the number of electrons leaving with or without an applied potential, the number of electrons leaving the surface of the alkali metal per second can be computed and compared with the number of atoms in the active layer. At 25° the negative current for potassium in darkness with zero potential is 0.283×10^{-12} amperes. And the positive current when the potassium is illuminated is 7.5×10^{-12} amperes. Since the positive current must be the difference between the current due to the electrons and that due to the positive particles, the current of electrons must be

$$7.5 \times 10^{-12} + 0.283 \times 10^{-12} = 7.783 \times 10^{-12} \text{ amperes,}$$

$$7.78 \times 10^{-12} \text{ amp.} = 7.78 \times 10^{-12} \text{ e.m. units.}$$

The number of electrons reaching the electrode per second is

$$\frac{7.78 \times 10^{-12}}{1.55 \times 10^{-20}} = 5.02 \times 10^7,$$

where 1.55×10^{-20} is the value of e in electromagnetic units.

The potassium was in the shape of a disk with a very nearly flat surface and a radius of about 1 cm.

Area of potassium = $\pi r^2 = 3.1416 \text{ cm}^2$.

Diameter of potassium atom = $4.74 \times 10^{-8} \text{ cm}$.

Cross-sectional area occupied by one potassium atom is

$$(4.74 \times 10^{-8})^2 = 2.25 \times 10^{-15} \text{ cm}^2.$$

$$\text{Number of atoms in one layer} = \frac{3.1416}{2.25 \times 10^{-15}} = 1.397 \times 10^{15}.$$

Ladenburg¹ has found that the thickness of the layer of metal effected by light is about 10^{-4} cm . Hence the number of layers of atoms of potassium in the active layer of the metal is given by

$$\frac{10^{-4}}{4.74 \times 10^{-8}} = 2.11 \times 10^3.$$

The total number of atoms in the active layer is

$$1.397 \times 10^{15} \times 2.11 \times 10^3 = 2.95 \times 10^{18}.$$

Ratio of number of atoms to number of electrons given off per second is

$$\frac{2.95 \times 10^{18}}{5.02 \times 10^7} = 5.88 \times 10^{10}.$$

That is, one atom out of 5.88×10^{10} gives out one electron per second. At this rate it would require 5.88×10^{10} seconds or 1,860 years for each atom to give out one electron. Of course the electron current can be considerably increased by using a more intense light and one of shorter wave-length. And then the ratio between the number of atoms and the number of electrons given off will be much decreased. But this computation serves to emphasize the fact that the number of atoms effected is exceedingly small. This may be due to one of two causes, or possibly to both: (1) only a very few of the atoms are in condition to give out electrons under the influence of light; (2) light has a structure and the energy is not uniformly distributed over the light wave front, but is concentrated along certain lines as suggested by the theory of J. J. Thomson² and amplified by Kunz.³ In this investigation there is some evidence in favor of the first explanation. A comparison of the curves for "potassium illuminated" and "in darkness" in Fig. 8 will show that as the temperature increases the effect of light on the current becomes less and less, until at 150° the curves for "potassium illuminated" and "in darkness" almost coincide. That is, at this temperature electrons are given off just as readily without light as with it when an electric field is applied, or the photo-electric action at this temperature is practically

¹ Ann. der Phys., 12, p. 558, 1903.

² Proc. of Camb. Phil. Soc., 14, pt. 4, p. 41, 1908.

³ Phys. Rev., 29, p. 212, 1909.

zero. Hence it appears that the electron current depends more upon the condition of the metal than of the incident light. If the number of electrons given off depended alone on the structure of light, the electron current should be much larger with light than without light for all temperatures.

At 150° with + 100 volts potential the current is about 1,000 times larger than at 25° and with zero potential. But even with this current the ratio between the number of atoms and the number of electrons given off is a large one, 5.88×10^7 . So that even at this temperature the number of atoms in condition to give off electrons is small.

Estimate of the Maximum Vapor Pressure Possible of Potassium.—When light is incident on a metal the electrons are given off with a velocity which can be obtained by the relation

$$Pe = \frac{1}{2}mv^2,$$

where e is the charge of the electron and P the maximum positive potential assumed by the metal in the photo-electric action. The maximum velocity of the electrons from potassium in this work when zero potential was applied is due to the shortest wave-length of visible light, about $\lambda 4,200$. D. W. Cornelius in this laboratory found the velocity due to this wave-length to be 6.21×10^7 cm. per second. When there is an electric field acting the velocity of the electrons increases beyond the initial velocity until the electron has sufficient velocity to produce an ion by collision. This relation is expressed by

$$W = Eel + \frac{1}{2}mv^2,$$

where W is the energy required to produce an ion, E the electric field, e the elementary electrical charge, m the mass of the electron, v the initial velocity and l the distance from the surface of the metal that the electron must go to gain sufficient velocity to produce ionization. The value of E can be taken from the curve at the point of departure from a straight line. From the curve for 25° , Fig. 8, we find the potential corresponding to this point of departure to be 350 volts. As the electrodes were 2 cm. apart, the value of E is

$$\frac{350}{2} = 175 \text{ volts per cm.,}$$

$$E = \frac{175}{300} = 0.583 \text{ e.s. units per cm.,}$$

$$e = 4.65 \times 10^{-10},$$

$$m = 8.7 \times 10^{-28},$$

$$v = 6.21 \times 10^7.$$

The value of W is taken at 1.58×10^{-11} ergs, which is the value obtained by Bishop¹ for hydrogen and also by Kemp in this laboratory. So far as has been investigated this value does not vary very much for different gases, so it will be assumed that it is a reasonable value for potassium vapor.

$$l = \frac{W - mv^2/2}{Ee} = \frac{1.58 \times 10^{-11} - \frac{8.7 \times 10^{-23}(6.21 \times 10^7)^2}{2}}{5.83 \times 10^{-1} \times 4.65 \times 10^{-10}} \\ = \frac{1.58 \times 10^{-11} - 0.169 \times 10^{-11}}{2.71 \times 10^{-10}} \\ = 5.2 \times 10^{-2} \text{ cm.}$$

Now the mean free path of the molecules of potassium vapor must be at least this great or there could be no ionization by collision at this field strength. Assuming this to be the minimum mean free path we can calculate the maximum number of molecules per cm³. The mean free path is given by

$$l = \frac{1}{\sqrt{2}n\sigma^2},$$

where n is the number of molecules per cubic centimeter and σ is the diameter of the molecule. We have then

$$n = \frac{1}{\sqrt{2}l\sigma^2} = \frac{1}{\sqrt{2} \times 5.2 \times 10^{-2} (4.74 \times 10^{-8})^2} \\ = 1.925 \times 10^{15}.$$

We may assume that at the temperature and pressure existing in the tube the potassium vapor acts as a perfect gas and so compute the pressure from

$$P = \frac{P_0 T n}{T_0 n_0},$$

where $P_0 = 760$ mm. pressure, $T_0 = 273^\circ$, n_0 the number of molecules in a cubic centimeter of gas at standard conditions and is equal to 2.72×10^{19} , T the absolute temperature of the vapor and n the number of molecules of the vapor per cubic centimeter. Hence

$$P = \frac{760 \times 298 \times 1.925 \times 10^{15}}{273 \times 2.72 \times 10^{19}} \\ = 0.0587 \text{ mm.}$$

¹ PHYS. REV., 33, p. 325, 1911.

This is of the order of magnitude that would be expected. The only other determination made of the vapor pressure of potassium is one by Keyes¹ from theoretical considerations for 400° which he gives as 1.4 mm.

This problem of conductivity in alkali vapors is by no means solved. One very important feature which should be investigated is the effect of ultraviolet light. Does ultraviolet light ionize the vapor? And is there ionization from the electrons given off from the metal? From the data obtained some prediction can be made on this last point. Taking the energy necessary to produce an ion to be 1.58×10^{-11} ergs we can compute the velocity that an electron should have to produce ions by collision.

$$W = \frac{1}{2}mv^2,$$

$$v^2 = \frac{2W}{m} = \frac{2 \times 1.58 \times 10^{-11}}{8.7 \times 10^{-28}} = 3.63 \times 10^{16},$$

$$v = 1.9 \times 10^8 \text{ cm. per second.}$$

If we assume that the velocity of the electrons in the photo-electric effect varies inversely as the wave-length, we have

$$v_2 = v_1 \frac{\lambda_1}{\lambda_2}.$$

Taking $v_1 = 6.21 \times 10^7$ for $\lambda_1 = 4,200$ and $\lambda_2 = 2,000$, which is about the shortest wave-length which can be obtained with a quartz spectrometer, we find

$$\begin{aligned} v_2 &= 6.21 \times 10^7 \frac{4200}{2000} \\ &= 1.304 \times 10^8. \end{aligned}$$

This is less than the value 1.9×10^8 computed above. But it is of the same order of magnitude and if the minimum energy to produce an ion is less than 1.58×10^{-11} ergs, it is entirely possible for the electrons set free from potassium by the action of the short wave-lengths of ultraviolet light to produce ionization by collision.

In some other work carried on by the author a potential of 5.85 volts was obtained from potassium illuminated by λ 2,100. This gives a velocity of 1.445×10^8 cm. per second which is some nearer the critical velocity to produce ionization.

These values would indicate that it would be well worth while to investigate this point.

¹ Jour. Am. Chem. Soc., XXXIV., p. 779.

SUMMARY.

1. This investigation has shown that when potassium vapor alone is between the electrodes of a tube there is no conductivity of the usual type at 25° and 50°; at 100° with a potential of 700 volts there is a current arising from ionization by the electric field. *Hence there is nothing of the character of a spontaneous ionization.*

2. When potassium was present in the tube but not in contact with the electrodes there is no conductivity of the usual type at 25° and 50°. At 100° there is a current due to particles given off from the potassium.

3. In a tube with two similar electrodes exhausted to the best degree possible there is a current of the order 10^{-12} amperes in a direction *opposite* to the electric field, which increases with the field. This phenomenon is effected by light and is decreased by the presence of potassium vapor.

4. The conductivity in potassium vapor when one electrode is potassium has been measured for temperatures up to 150°.

5. The conductivity in caesium vapor when one electrode is caesium has been measured for temperatures up to 100°.

6. Woodrow's observations on the emission of positive particles from alkali metals have been confirmed and for temperatures above 50° found to be large compared to the electron current.

7. The ratio between the number of electrons given off per second and the number of atoms present has been found for a given source of light to be $\frac{1}{5.88 \times 10^{10}}$.

8. At 150° light has practically no effect on the emission of electrons from potassium.

9. By comparing the currents for different temperatures in tube No. 2, with potassium electrode, it is found that the greatest relative effect of light on the emission of electrons is at 25°.

10. The maximum vapor pressure possible for potassium at 25° has been found to be 0.0587 mm.

The author wishes to express his indebtedness to Professor A. P. Carman and the department of physics for the facilities for this investigation, and to Professor Jakob Kunz who suggested the problem and has given many valuable suggestions.

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May, 1912.

HEATS OF DILUTION.

BY WILLIAM FRANCIS MAGIE.

1. *Formulas for Heats of Dilution.*—Consider a system consisting of a solution made up by dissolving one gram-molecule of a salt in N gram-molecules of water and an additional mass of water. It is known that if the solution is diluted by the transfer of some of the water to it the heat capacity of the system diminishes. It is also known that the rate at which heat is evolved as the volume of the solution is thus increased (the heat of dilution) differs remarkably at different temperatures. These effects are connected by the formula

$$\frac{dl}{d\theta} = - \frac{dH}{dv}, \quad (a)$$

in which l represents the heat of dilution, taken positive when heat has to be removed from the solution to keep its temperature constant as dilution proceeds, H , the heat capacity of the system, θ , the absolute temperature, and v , the volume of the solution. The proof of this formula is obtained by equating the quantities of heat which enter the system when it is transferred from an initial state at a certain temperature θ , to a final state at a slightly higher temperature $\theta + d\theta$, in two ways: (1) by raising the temperature of the system from the initial to the final value and then transferring the volume dv from the water to the solution at the final temperature; (2) by transferring the volume dv from the water to the solution at the initial temperature and then raising the temperature of the system from the initial to the final value; on the assumption that the external work done by the changes of volume which accompany dilution may be neglected.¹

The experiments of Teudt² have shown that dH/dv is independent of the temperature, within the range of temperature open to experiment. Representing dH/dv by a , a function of the volume but not of the temperature, and integrating over a finite temperature range we have

$$l_2 - l_1 = -a(\theta_2 - \theta_1) \quad (b)$$

or generally

$$l = -a\theta + e, \quad (c)$$

where e is a function of the volume but not of the temperature.

¹ Thomsen, *Thermodynamische Untersuchungen*, Vol. 1, p. 65.

² *Inaug. Diss. Erlangen: Beiblätter. XXIV., p. 1104 (1900).*

The heats of dilution are determined by starting with an initial concentration of the solution and measuring the amounts of heat, L_0 , evolved as the solution is diluted by the transfer to it of the volume v of water (practically by the increase of the volume of the solution by v). If the values of L thus obtained are plotted against the values of v and a curve drawn through the points thus given, the tangent to this curve at any point is the value of l for the corresponding concentration.

We may thus write formula (b) in the form

$$\frac{dL}{dv_2} - \frac{dL}{dv_1} = - \frac{dH}{dv} (\theta_2 - \theta_1) \quad (d)$$

and integrate it with respect to the volume, since the changes of temperature caused by dilution are so small as not to affect the values of L found in the experiments. We obtain

$$(L_v - L_0)_2 - (L_v - L_0)_1 = - (H_v - H_0)(\theta_2 - \theta_1), \quad (e)$$

in which the subscripts v and 0 refer to the final and initial concentrations.

It is more convenient to plot the curves and to express the formulas in terms of the concentrations than in terms of the volumes. This can be done when the concentration is defined in any arbitrary way, for the concentration c is practically independent of the temperature, so that we may write formula (d) in the form

$$\left(\frac{dL}{dc_2} - \frac{dL}{dc_1} \right) \frac{dc}{dv} = - \frac{dH}{dc} \cdot \frac{dc}{dv} (\theta_2 - \theta_1)$$

and setting $dL/dc = \lambda$, $dH/dc = \alpha$, we have

$$\lambda_2 - \lambda_1 = - \alpha (\theta_2 - \theta_1) \quad (f)$$

and similarly

$$\lambda = - \alpha \theta + \epsilon. \quad (g)$$

The dc/dv disappears from the formulas and we can define c as we please. In the experiments which are to be described c was set equal to $100/N$, where N is the number of gram-molecules of water containing one gram-molecule of the salt. The quantities α and ϵ are functions of the concentration but not of the temperature.

2. *Method of Observation.*—The experiments of Thomsen,¹ from which he concluded that the temperature coefficient of the heat of dilution is always positive, do not afford a demonstration that the foregoing formulas are valid. Since their validity depends upon the assumption that the

¹ Therm. Unters., Vol. I, p. 80.

quantities a or α are independent of the temperature, an experimental verification of them will confirm this assumption. I have undertaken to obtain heats of dilution at different temperatures for solutions of certain salts for which the heat capacities were accurately known. The confirmation of the formulas is still not altogether satisfactory, but I am convinced that this is due to the experimental inaccuracies of the method employed.

The solution to be diluted was held in a silver calorimeter vessel, with a capacity of over 1,200 c.c. Usually so much of the solution was taken that the water contained in it weighed 500 grs. This was diluted with 500 grs. of water, run in from another silver vessel, after the initial temperatures of the solution and of the water had been determined. The thermometers used were made by Fuess, and were divided directly to fiftieths of a degree, so that they could be read by estimation to thousandths of a degree. It generally happened that neither the initial nor the final temperatures found were fixed, but changed progressively and quite uniformly as the liquids were stirred. In all cases several readings were made at intervals of a minute and an estimate was made, from the readings at the ends of the series and the rate of change of temperature determined, of the temperatures of the liquids, just before and just after mixing.

Trials with water in both vessels and the observations to determine the water equivalent of the vessel in which the liquids were mixed showed that the percentage error of the observations made in this way was not important when the changes of the temperature caused by the dilution was considerable, but the uncertainty of the measurements when the changes were of only a few thousandths of a degree, as was often the case with the more dilute solutions, was so great that the results for these cases are not reliable.

3. *Results.*—The results obtained are contained in the following table. The salts used were the chlorides of sodium, potassium, ammonium, barium, and strontium. The initial solution contained 25 gram-molecules of water to each gram-molecule or each half gram-molecule of salt as indicated in the tables. The initial concentration, on the convention adopted that $c = 100/N$, is therefore 4. In the first column of each table are given the concentrations reached by dilution from this initial concentration. In the other columns, under the values of the temperatures at which the experiments were carried out, are given the amounts of heat in gram-calories evolved (+) or absorbed (−) on the dilution from the initial to the indicated concentration of a solution containing one gram-molecule of the salt.

c	NaCl.		KCl.		NH ₄ Cl.	
	26.4°	16.8°	21°	3°	22.5°	3.7°
2	-157	-218	-165	-288	-20	-94
1	-239	-335	-248	-460	-20	-145
$\frac{1}{2}$	-267	-379	-285	-555	-11	-169

c	$\frac{1}{2}$ BaCl ₂ .			$\frac{1}{2}$ SrCl ₂ .		
	24.5°	16.1°	7.1°	23.3°	16°	4.3°
2	+ 7	-42	-109	+ 35	- 3.4	- 75
1	+26	-51	-158	+ 66	+ 0.8	-117
$\frac{1}{2}$	+50	-39	-173	+102	+22.5	-144
$\frac{1}{4}$	+85		-164	+134		-122

4. *Test of Integral Formula.*—The degree to which these results conform to formula (e) may be shown by one or two examples. The L_0 of the formula equals 0, and the two terms on the left are the numbers given for two temperatures for the same final concentration. The values of H are obtained from the tables in my paper on the Specific Heat of Solutions, V.,¹ using the calculated rather than the observed results, and obtaining the initial value, when it is wanting, by interpolation. It is to be observed that the H of these tables is the heat capacity of the solution only and does not include that of the water needed to complete the system of which the H of formula (e) is the heat capacity.

In the case of sodium chloride we find that the successive values of $\frac{(L_2 - L_0)_2 - (L_2 - L_0)_1}{\theta_2 - \theta_1}$ are 6.3, 10, and 11.7, while starting with 448.2 for the heat capacity of a solution of sodium chloride for which $N = 25$, obtained by interpolation, we have for $-(H_2 - H_0)$ the corresponding numbers 6, 10.2, 13.8. In the case of barium chloride we find for the successive ratios, using the extreme temperatures, the values 6.6, 10.5, 12.8, 14.3, while starting with 430.6 for the heat capacity of a solution for which $N = 25$, obtained by extrapolation, we have for $-(H_2 - H_0)$ the corresponding numbers 6.2, 10, 13.7, 16.6. The results for the other substances show about the same sort of agreement, except in the case of ammonium chloride, in which there seems to be some systematic error.

5. *Test of Differential Formula.*—When the total heats of dilution (L) given in the tables are plotted against the corresponding concentrations (c), and the heat capacities (H) are plotted against the same concentrations, the tangents to the curves thus determined for the concentration furnish values of λ and α respectively. These values conform to the

¹ This REVIEW, Vol. XXV., p. 171 (1907).

relation expressed in formula (f). In all cases at the low temperatures, and in some cases at the higher temperatures, the L_c curve is nearly a straight line, as has been shown by Bishop¹ for solutions of KCl and of other salts at 25°. Since the values of α are never constant for all concentrations the L_c curve is really straight only by exception.

The curves for solutions of barium chloride furnish an illustration of the general form of these curves (Fig. 1). The tangent at concentration 1 to the curve for 24.5° is - 37; that to the curve for 7.1° is + 38. The difference of these tangents divided by the difference of the temperatures is - 4.3, and this is equal to the negative value of the tangent at the same concentration to the curve of heat capacities, as it should be to conform to formula (f). By calculation from these data we find that for the same concentration the temperature at which $\lambda = 0$ is 16°. The curve for 16° confirms this conclusion. A similar transition of λ from negative to positive values was found at 16° for concentration 2 in the case of strontium chloride, and at 22.3° for concentration 1.5 in the case of ammonium chloride. All these relations confirm the experimental results of Teudt, according to which the quantities a and α are independent of the temperature.

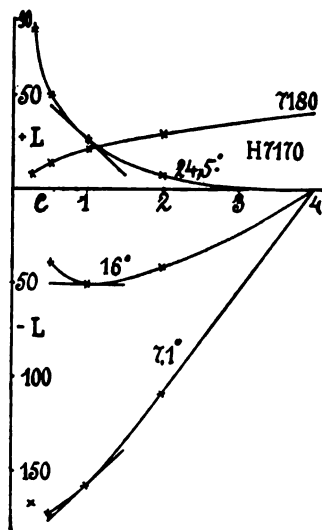


Fig. 1.

6. *Explanation of Heats of Dilution on the Association Theory of Solutions.*—The most striking thing about these heats of dilution is their very large proportionate change with change of temperature. The reason for this will be seen on an examination of formula (c). This formula shows that the heat of dilution is the sum of two terms of opposite sign, one of which is proportional to the absolute temperature, the other, independent of the temperature. The term $-a\theta$ is always positive, for the heat capacity of the system diminishes as the volume of the solution increases, so that a is negative. It corresponds to an evolution of heat. The other term ϵ is negative and corresponds to an absorption of heat. These terms are both large and a change in one of them, proportional to the change in the absolute temperature, may result in a change in the difference between them which is large in comparison with that difference.

¹ This REVIEW, Vol. XXVI., p. 169 (1908).

The evolution of heat to which the term $-a\theta$ corresponds can be explained by considering the change in the number of degrees of freedom in the solution which results from dilution. It has been shown¹ that the rate of change of heat capacity represented by the factor a is proportional to the rate of change of dissociation, and that there must be some interaction between the molecules and ions of the solute and the molecules of water which brings about a diminution in the number of degrees of freedom of the water. I have used the word "interaction" instead of the less general word "association," which was used in the paper referred to, in deference to a suggestion made by Professor G. N. Lewis, at the Cambridge meeting of the American Physical Society, that the effect might be produced by changes in the state of aggregation of the molecules of water without there being any immediate association between the aggregations of water molecules and the solute. For the purpose of the present explanation it is a matter of indifference which view is taken. A study of the constants of the formula which represents the heat capacity of a solution shows that usually the number of degrees of freedom of the undissociated molecules of the solute and of the water affected by them is increased by solution, and that always the number of degrees of freedom of the ions and of the water affected by them is decreased. As dilution proceeds, therefore, and the molecules of the solution break up into ions, the number of degrees of freedom of the solution diminishes both by the removal of molecules from the solution and by the introduction of new ions. The change in the number of degrees of freedom is proportional to the change in the amount of dissociation. The energy which has been associated with the degrees of freedom thus removed from the solution is released in the solution and appears in the form of heat.

At the temperatures at which the experiments have been made and at which the formula (c) is valid, the energy associated with a degree of freedom is proportional to the absolute temperature. An evolution of heat proportional to the change in dissociation and to the absolute temperature is thus accounted for.

The heat absorbed, to which the negative term ϵ corresponds, is nearly equal in amount at all concentrations to the heat evolved. It also is thus approximately proportional to the rate of change of dissociation. It may involve the heat of dissociation, corresponding to the work done in splitting up the molecules into ions. It may also involve the heat developed by the internal work done as the dilution proceeds and the volume of the system diminishes. Neither of these effects would depend

¹ Magie, "Specific Heat of Solutions," V., *this REVIEW*, Vol. XXV., p. 171 (1907).

primarily upon the temperature. The values obtained from the values of ϵ for the heat absorbed at a given concentration by the complete dissociation of a gram-molecule of the salt are of the same order of magnitude as the heat evolved by combination of the atoms to form a gram-molecule of the salt. Thus in the case of sodium chloride the values of the heat absorbed by the complete dissociation of a gram-molecule at concentrations 2, 1, $\frac{1}{2}$ are 22,700, 21,800, 21,200 gram-calories respectively. The heat of combination of a gram-molecule of sodium chloride is given in the Smithsonian Physical Tables as 97,000 gram-calories. By calculating from the heat absorbed by the dilution of a solution of potassium chloride from concentration 4 to concentration 2 we find the heat absorbed by the complete dissociation of a gram-molecule equal to 40,000 gram-calories, while the heat of combination of a gram-molecule of potassium chloride is 105,000 gram-calories. It seems reasonable to conclude that the term ϵ of the formula contains, as an important part of it, the heat of dissociation.

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THE RELATION OF OSMOTIC PRESSURE TO TEMPERATURE.

BY WILLIAM FRANCIS MAGIE.

1. *Formula for Osmotic Pressure.*—By a thermodynamic argument¹ it is easy to establish the formula,

$$\frac{d^2p}{d\theta^2} = \frac{a}{\theta}, \quad (a)$$

in which p represents the osmotic pressure, θ , the absolute temperature, and a , the rate at which the heat capacity of a solution changes as its volume increases. We shall consider the quantity a independent of the temperature (*vide* paper on Heats of Dilution in the present number of this REVIEW). Integrating on this assumption we obtain

$$p = a\theta(\log \theta - 1) + b\theta + e, \quad (b)$$

in which b and e are quantities which are functions of the concentration of the solution, but are independent of the temperature. This formula expresses the proposed relation of the osmotic pressure to the temperature.

2. *Deduction of Formula for Heat of Dilution.*—When a solution is diluted by the addition of the small volume w of solvent, the heat which must be abstracted to keep the temperature constant is the measure of the change of internal energy E brought about by dilution, since the external work done in consequence of the change of volume is so small as to be negligible. This heat, which is proportional to the heat of dilution, is reckoned positive if the change in the internal energy is negative. The change of internal energy is expressed in terms of the change of free energy F by the formula

$$E = F - \theta \frac{dF}{d\theta}.$$

The change of free energy is measured by the work $-pw$ done by the osmotic pressure p when the original state of the solution and solvent is restored by separating the volume w of solvent from the solution in a reversible way by the use of a semi-permeable membrane. Setting $F = -pw$ and using the formula (b) we obtain

$$E = w(a\theta - e).$$

¹ Magie, The Specific Heat of Solutions, this REVIEW, Vol. IX., p. 65 (1899).

The negative value of E/w represents the heat of dilution l . We thus obtain the formula for the heat of dilution

$$l = -a\theta + e \quad (c)$$

which was obtained in the paper referred to from other considerations.

3. *Test of Formula for Osmotic Pressure.*—The quantity a is known from observations of the heat capacity, the quantity e can be calculated from observations of the heat of dilution, and then the quantity b calculated from observations of the osmotic pressure or of other properties of solutions which depend on the osmotic pressure. The observations which are best adapted to furnish useful values of p and therefore of b are those of the freezing points and the boiling points of the solutions.

The only solutions for which I have been able to obtain the requisite data to test the formula are those of sodium chloride in water. The observations of Thomsen¹ furnish the heat capacities, those of Kahlenberg² the freezing and boiling points, and my own observations³ the heats of dilution. In effecting the calculations it is convenient to express the quantity a in gram-calories per cubic centimeter, instead of in mechanical units. The osmotic pressure p is reduced to a consistent unit by dividing its value in dynes per square centimeter by Joule's equivalent. From the freezing points for sodium chloride solutions given in Kahlenberg's table (*l. c.*, p. 353) the corresponding osmotic pressures were calculated. They were then plotted against the number N of gram-molecules of water which contained one gram-molecule of salt in each case and a curve drawn through the points thus determined from which the osmotic pressure at 0° C. and at the standard concentrations could be read off. A similar calculation from the boiling points given in Kahlenberg's table (Series 3) (*l. c.*, p. 362), which furnished the largest number of available observations made at one time, established another curve for the determination of osmotic pressure at 100° C.

The necessary values of a are obtained from the tangents to the curve of heat capacities plotted against the concentrations, as described in my paper on "Heats of Dilution," by multiplying these tangents by the reduction factor $100/18N^2$. This factor is the value of dc/dv on the conventions that $c = 100/N$ and that $v = 18N$, which is sufficiently accurate at all concentrations, in view of the uncertainties in the observations. The values of l are obtained from the tangents to the curve of heats of dilution plotted against the concentrations, as described in the paper just referred to, by the use of the same reduction factor.

¹ Magie, "Specific Heats of Solutions," V., *this REVIEW*, Vol. XXV., p. 171 (1907).

² Jour. Phys. Chem., Vol. V., p. 339.

³ "Heats of Dilution," *this REVIEW*, the present number.

The following table contains the values adopted for the concentrations 2, 1, $\frac{1}{2}$, of a , of p_0 at the freezing point, and of l at 26.4°C ., and the values calculated from them of the quantities b and e and of p_{100} at the boiling point.

c	a	p_0	l	b	e	p_{100}
2	0.00733	1.105	-0.1820	0.04656	-2.378	1.532
1	0.002833	0.550	-0.0361	0.01831	-0.8843	0.749
$\frac{1}{2}$	0.001139	0.280	-0.00332	0.007537	-0.3443	0.377

The values calculated for the osmotic pressures at the boiling point lie very exactly on the curve plotted from Kahlenberg's observations of the boiling points. They therefore furnish a satisfactory confirmation of the formula connecting the osmotic pressure with the temperature and thus incidentally confirm the assumption that the quantity a is independent of the temperature.

From the values adopted for a and e we find that the transition temperatures at which l passes from negative to positive values should be 51° , 39° , 29°C . for the concentrations 2, 1, $\frac{1}{2}$ respectively.

4. *Test by von Babo's Law.*—Another test of the formula is obtained by calculating from it the ratio of the vapor pressure over water to the vapor pressure over a solution of a chosen concentration at different temperatures. According to von Babo's law this ratio should be independent of the temperatures. The ratios calculated for a sodium chloride solution of concentration 1 for temperature intervals of 20° , beginning with the freezing point and ending with the boiling point, are 1.01831, 1.01852, 1.01847, 1.01851, 1.01849, 1.01825. The ratio comes out appreciably constant. The observations of Emden¹ on solutions of sodium chloride are in accord with this result.²

The vapor pressures of solutions should conform to von Babo's law when the osmotic pressure obeys Gay-Lussac's law, that is, when the osmotic pressure is proportional to the absolute temperature. The following table shows how closely the osmotic pressure calculated from the

¹ Wied. Ann., XXXI., p. 145.

² It is worthy of notice that the value 1.01831, obtained from Kahlenberg's freezing points for the ratio of the vapor pressures over pure water and over a sodium chloride solution of concentration 1, is confirmed by the direct and very careful observations of Dieterici (Wied. Ann., LXVII., p. 859). When Dieterici's observed ratios are plotted against the concentrations they determine a straight line which passes through the point (1, 0). From this line we find for the ratio above the value 1.0185. Not only so, but all of the ratios obtained for different concentrations from Kahlenberg's freezing points lie very accurately on the same line. The agreement of the results of these two sets of observations is a striking confirmation of the accuracy of both of them.

formula for various temperatures in the case of a solution of sodium chloride of concentration 1 agrees with those calculated by Gay-Lussac's law.

θ	273	293	313	333	353	373
P, formula.....	0.550	0.597	0.636	0.678	0.718	0.749
P, G. L. law.....	0.550	0.590	0.630	0.671	0.711	0.751

The agreement is very close in this case. As it depends on the relative values of the coefficients in the formula, it cannot be expected to be equally close in all cases.

If the formula for osmotic pressure is valid the temperature coefficient of the osmotic pressure becomes

$$\frac{1}{p} \frac{dp}{d\theta} = \frac{1}{\theta} \frac{p - l}{p}.$$

It becomes $1/\theta$ and the osmotic pressure conforms strictly to Gay-Lussac's law, when the temperature is that for which $l = 0$. For the sodium chloride solution of concentration 1, this coefficient equals 0.00438 at 0° C. and 0.00205 at 100° C.

5. *Origin of Osmotic Pressure.*—The first and last terms in the formula for the osmotic pressure are manifestly terms which do not depend upon the motions of the particles of the solute. They represent rates of change of energy with change of volume of the solution, and have been explained in my paper on "Heats of Dilution" by the assumption of forces acting between the molecules and ions of the solute and the molecules of the water. Their appearance in the formula for osmotic pressure indicates that the osmotic pressure arises from these forces and does not primarily depend upon the motions of the particles of the solute.

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ON CHARACTERISTIC ATOMIC CHARGES AND RESULTANT MOLECULAR CHARGES.

BY FERNANDO SANFORD.

IN a paper published a year ago under the caption "On Positive Atomic Charges"¹ I gave reasons for thinking that the positive sub-atoms of the elements have characteristic charges differing from each other and approximately proportional to the square roots of their atomic weights. On page 516 of that article I gave relative values for a few of these charges calculated in part from the velocity of ions in electrolysis and partly from data given by Richardson and Hulbirt in *Phil. Mag.*, XX., 545, Oct., 1910.

In a paper "On Rays of Positive Electricity" in *Phil. Mag.*, XX., 752, Oct., 1910, Sir J. J. Thomson gives values of e/m for a number of different positive particles found in the canal rays in an exhausted tube, and names these particles on the assumption that they carry positive charges equal to one or more unit hydrogen charges. In the paper referred to Thomson observed a number of fluorescent patches on his screen, one of which, a central patch called *o*, was not deflected by the electric or magnetic fields, while the others, named respectively *a*, *b*, *c*, *d*, *e* and *f*, showed deflection due to positive charges. The values of e/m calculated for the particles which produced these patches are given as follows: *a*, 65; *b*, 830; *c*, 2,900; *d*, 5,800; *e*, 11,600; *f*, 12,400. Thomson identifies these particles as follows: *a*, due to mercury vapor; *b*, due to atoms of oxygen or nitrogen; *c*, due to atoms of helium with one positive charge; *d*, due to hydrogen molecules with one surplus atomic charge; *e*, due to hydrogen atoms; *f*, due to secondary rays of some kind whose velocity is independent of the voltage in the tube.

In a paper by Professor C. T. Knipp on "Rays of Positive Electricity from the Wehnelt Cathode," *Phil. Mag.*, XXII., 926, Dec., 1911, are given the values of e/m for a number of positive particles sent off from a hot platinum cathode on which is a spot of CaO. Knipp finds values of e/m which agree quite closely with those determined for canal rays from the anode by Sir J. J. Thomson.

Knipp's highest value of e/m is 9,400, and he identifies this with the hydrogen atom. His next value is $e/m = 4,960$, and this particle he

¹ *PHYSICAL REVIEW*, XXXII., 512, May, 1911.

identifies, after Thomson, as a hydrogen molecule with an excess positive charge equal to that on a single hydrogen atom. Thomson's values for the corresponding particles in the canal rays are 11,600 and 5,800.

Since it is difficult to see how a gas molecule could acquire an excess positive atomic charge without annexing the positive sub-atom which belongs with the charge, or could part with an electron without dissociating into atoms, it seems permissible to look for some other positive particle which may have the given value for e/m . Rutherford and Geiger found for the alpha-particle a value of $e/m = 4,800$ to 5,700, and since Knipp's value lies between these limits it seems at least as permissible to assume that there were alpha-particles in the tubes as to assume the existence of a gas molecule with an atomic charge.

In a paper entitled "The Significance of the Periodic Law," Journ. Am. Chem. Soc., XXXIII., 1349, Aug., 1911, I have given, as I believe, valid reasons for expecting alpha-particles to be given off from the anode in an exhausted tube. If such should prove to be the case, the tube should, in both investigations, contain particles having a value of e/m one half as great as the alpha-particle, and other particles which would not be deflected in the electric or magnetic field; for the alpha-particles would in both cases be compelled to traverse a region containing free electrons, and many of them, if not most of them, would take up one or two electrons in their flight. Those taking up one would have their value of e/m reduced by one half, and those taking up two would become electrically neutral helium molecules and be undeflected in the electric and magnetic fields.

Particles having one half the above value of e/m were numerous in Thomson's tube, but seem to have been rarer in Knipp's; but he gives several measurements on them with an average value of $e/m = 2,380$. Thomson identifies these particles as here indicated, viz., as helium atoms with one positive charge, which is the same as alpha-particles with one electron. The mean of Thomson's and Knipp's values of e/m for these particles is 2,640, while the mean of alpha-particles plus one electron should be from Rutherford and Geiger's determinations 2,625.

One spot on Knipp's plates which did not appear on Thomson's was due to particles having an average value of $e/m = 1,700$. This particle Knipp identifies as the carbon atom with two positive atomic charges. In a later paper in PHYSICAL REVIEW, XXXIV., 215, March, 1912, Knipp gives data on two plates made with a much higher accelerating field and a higher discharge potential, and finds spots produced by only two different kinds of particles, one with an average value of $e/m = 4,830$, which again is the correct value for the alpha-particle, and one giving

$e/m = 1,860$, and which he identifies as the one for which he determined the value 1,700 in his former plates. They are accordingly identified as carbon atoms carrying two positive charges, though in this case their value of e/m should equal one sixth that of the hydrogen atom, *i. e.*, 1,570 instead of 1,700, in the plates upon which both appeared.

Since the Wehnelt cathode was red hot in Knipp's experiments, it should have given off positive metallic ions of calcium and platinum. In my paper referred to above the relative values of the positive charges on hydrogen and calcium ions as calculated from their velocity in elec-

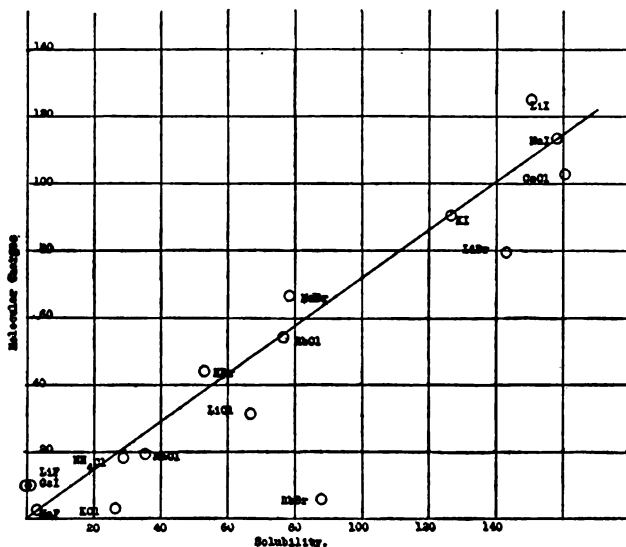


Fig. 1.

trollysis is as 11 to 79. Their relative values of e/m should accordingly be as $11 : 79/40 = 5.5 : 1$, approximately. Using Knipp's value for hydrogen, $e/m = 9,400$, the value of e/m for calcium should accordingly be 1,700, as it was found to be in the same experiment. This seems to indicate that in the experiments described in the second paper Knipp found only alpha-particles and calcium ions in his tube.

In his first paper Knipp also gives the, so called, electric atomic weight of another set of particles, which varies from 21.7 to 23.2. Richardson and Hulbirt give for the electric atomic weight of platinum 25 and 25.7, respectively, for two sets of observations. Knipp does not name these particles.

In the curve plotted by me in the article referred to above¹ showing

¹ Also given in Plate I. of "A Physical Theory of Electrification," Leland Stanford Jr. Publications, University Series, No. 6, May 15, 1911.

the relation of the atomic charges as there calculated to the square roots of their atomic weights, it will be seen that the relative charge of platinum as calculated from Richardson and Hulbirt's data is apparently too large to correspond with its atomic weight. If the value of its electric atomic weight taken from Knipp's data be substituted, the relative charge of platinum in my table on p. 516 will be 431 instead of 474, which would be in much better agreement with the law there suggested.

If it be granted, as here suggested, that in Thomson's and Knipp's tubes were both alpha-particles and hydrogen atoms, it becomes a simple matter to calculate the absolute charges of the elements referred to in

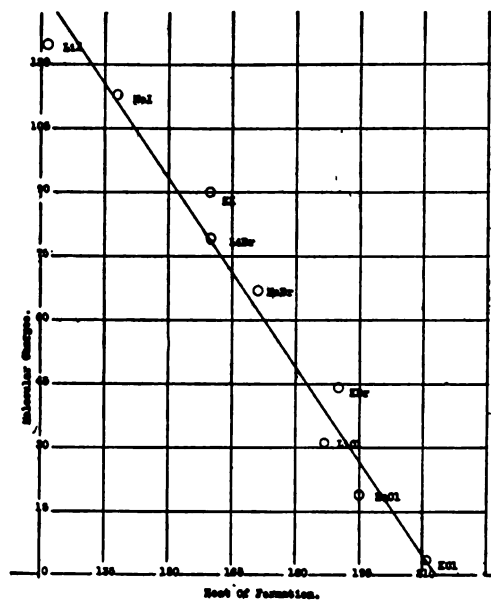


Fig. 2.

my former paper from the relative charges there given. Starting from the alpha-particle, which is the only gaseous ion whose charge has been directly measured, we can calculate the charge of the hydrogen ion from Thomson's and Knipp's data. Assuming the atomic weight of the alpha-particle as twice that of the hydrogen atom, its relative charge in my table would be 22 from Thomson's paper and 23.4 from Knipp's. Substituting its absolute charge for this value, the absolute charges of the other atoms are easily calculated.

While it is frankly admitted that the existence of these characteristic charges is still only a hypothesis, the same must be said of the assumption

that all positive ions carry simple multiples of the hydrogen charge. In the papers referred to above I have shown that these numbers calculated from the velocity of the ions in electrolysis are related to the atomic weight by the same laws as are a considerable number of other atomic constants which may reasonably be supposed to depend upon the electric condition of the atom. Among other properties, it was shown that the solubility of the monatomic gases in water is related to the square roots of the atomic weights as are the electric charges here assumed. It would seem to follow that other molecules should likewise be soluble in water in proportion to the difference between their molecular charges and the charge of the water molecules.

We are in a position to test this hypothesis in the case of some of the salts of the alkali metals and the halogen acids. If the molecules of these salts before dissociation in water were composed merely of the two ions which seem to take part in electrolysis, it is possible on the assumption of characteristic ionic charges to calculate the charge of the molecule before dissociation. In the following table these data are given for all the salts of this kind whose solubilities in water at zero are given in Seidell's Solubilities of Inorganic and Organic Substances, edition of 1907. The charges are calculated from the velocities of the ions in electrolysis, and are based upon the assumption that the charge of the hydrogen ion is 4.48×10^{-10} . They have all been multiplied by 10^{10} .

In the sixth column of the table is given the heat of formation of such of these compounds as are given in Julius Thomsen's Thermo-Chemistry.

Ion.	Charge.	Molecule.	Charge.	Solubility.	Heat of Formation.
F	- 13.6	LiF	- 9.6	.27	—
Cl	- 35.2	LiCl	- 31.	67.	187
Br	- 82.8	LiBr	- 79.	143.	160
I	-129.	LiI	-125.	151.	122
Li	+ 3.96	NaF	+ 2.	4.	—
Na	+ 15.8	NaCl	- 19.	35.6	195
K	+ 38.8	NaBr	- 67.	79.5	171
Rb	+ 89.	NaI	-113.	158.7	138
Cs	+138.	KCl	+ 3.6	27.1	211
NH ₄	+ 17.7	KBr	- 44.	53.5	190
		KI	- 90.	127.5	160
		RbCl	+ 54.	77.	
		RbBr	+ 6.	89.6	
		CsCl	+103.	161.4	
		CsI	+ 9.	1.	
		NH ₄ Cl	- 17.5	29.4	

In order that the relation of the solubility and the heat of formation to these resultant molecular charges may be seen more clearly, two curves are given in which the resultant molecular charges are plotted against the solubilities and the heats of formation, respectively. While there are marked exceptions, of which RbBr is the most striking example, it will be seen that the solubilities are, at least roughly, proportional to the resultant molecular charges, and that the heats of formation are still more closely related to them.

Whatever explanation may be given for these relations, it seems unquestionable that what are here called resultant molecular charges are actual physical constants which are in some manner closely related to cohesion.

It, of course, follows from the above relations that the solubilities of these salts in water bear a definite relation to their heats of formation, as may be seen by plotting one of them against the other.

STANFORD UNIVERSITY,
June, 1912.

COMPARATIVE STUDIES OF MAGNETIC PHENOMENA. III.

MAGNETIC INDUCTION IN A GROUP OF OBLATE SPHEROIDS OF SOFT IRON.¹

BY S. R. WILLIAMS.

IN the relation² between the Joule magnetostrictive effect and the magnetic induction in the same specimen of steel, I have pointed out that the maximum elongation of the steel rod in the Joule effect and the knee of the induction curve occur at the same magnetic field strengths. If the change in length of the steel rod is due to the orientation of elongated elementary magnets³ then the orientation of these same elements, possessing the property of permeability, will produce in their turning changes in the magnetic induction. That is saying that the variation of B with H in the ordinary magnetization curve is not alone due to the intrinsic value of the permeability of the elementary magnets but is influenced by the orientation of those same elementary magnets. It is a well-known fact that an elongated piece of iron will turn so that its greatest length is parallel to the field imposed upon it. The converse of this must hold that if an elongated piece of iron is turned in the magnetic field, the magnetic flux will be changed in the piece of iron because a piece of ferro-magnetic substance sets itself in a magnetic field so that the magnetic resistance is a minimum.⁴

The occurrence of the maximum elongation and the knee of the induction curve at the same field strength emphasizes the point of view that there is some factor in the specimen of iron being investigated which simultaneously changes the length of the specimen and also varies the induction. This factor, I believe, is the elementary magnet. In this series of comparative studies on magnetic phenomena I have been trying to find some point of view that will unify our present knowledge of magnetic phenomena and so far have found the idea of an elementary magnet, described in a previous paper, as being exceedingly helpful. No stress is laid on the theory only in so far as it leads to new facts.⁵

¹ Read by title at the Cambridge Meeting of the Amer. Phys. Soc., Apr., 1912.

² *PHYS. REV.*, p. 258, Vol. 34, Apr., 1912.

³ *PHYS. REV.*, p. 40, Vol. 34, Jan., 1912.

⁴ Starke, *Experimentelle Elektrizitätslehre*, p. 75, 1904.

⁵ Rücker, *Pres. Address*, B. A., Glasgow, 1909. See Mellor, *Higher Mathematics for Students of Chemistry and Physics*, p. 365.

To give added support to the suggestions offered above, the magnetic induction in a group of oblate spheroids of soft Swedish iron was studied to see if their orientation produced an effect on the magnetic flux. Oblate spheroids were chosen as the form for the "elongated particles" because this was the shape ascribed to the nucleus of the model of the elementary magnet.

It is evident that if a definite number of these oblate spheroids were laid with the minor axes on the same straight line and touching each other that their total length would be less than though the same number were laid with major axes on a similar line. A change in length of such a group, due to their orientation, is obvious. The magnetic induction, however, may be considered at greater length and this paper has for its object the study of the magnetic induction in a group of twenty-seven oblate spheroids of soft Swedish iron, arranged in the form of a cube with nine spheroids on a side. The minor axes measured one centimeter and the major one and one half centimeters.

Miss Laura Anderegg, a graduate student in the department, carried out the measurements in the following way:

MISS ANDEREGG'S EXPERIMENTS.

The spheroids were laid up in the form of a cube as shown in Fig. 1 and paraffine poured in around them. This kept the spheroids in a definite position while testing the magnetic induction. This cube of spheroids was placed inside of a small secondary coil whose dimensions were, length 10.7 cm., cross-section 6.5 cm. square and the number of turns, 170. This coil was connected in series with a ballistic galvanometer and with its core of spheroids placed at the center of the large solenoid described in a former paper.¹ The induction was studied by the ordinary ballistic method employed in testing specimens of iron, steel, etc. As only relative values were wanted the deflections of the ballistic galvanometer were used to represent the total flux through the secondary.



Fig. 1.

In Fig. 2 is shown the relative values for the magnetic flux, (1) when the spheroids were removed from the secondary, (2) when the spheroids were in the secondary and the equatorial planes *normal* to the imposed magnetic field and (3) when the spheroids were in the secondary but the equatorial planes *parallel* to the field. In both curves 2 and 3 there was the same amount of ferromagnetic substance present but the orientation of the elementary units with respect to the field produced a comparatively

¹ PHYS. REV., p. 41, Vol. 34, Jan., 1912.

large change in the magnetic flux; when the major axes were *parallel* to the magnetic field the flux was greater than when the major axes were *normal*. For curves 2 and 3, Fig. 2, the spheroids were laid so that they touched each other in two directions at right angles, as shown

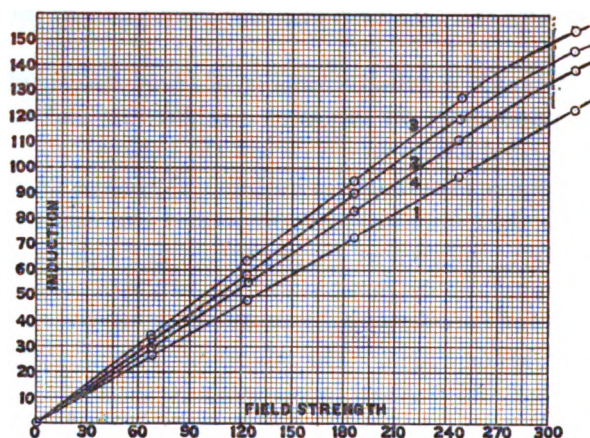


Fig. 2.

in Fig. 1. This made the length of the group shorter in one direction than in the other. When this shorter length was parallel to the imposed magnetic field there was less magnetic flux through the group than when the greater length was parallel. To make the outside dimensions of the group the same in all directions, the three layers shown in Fig. 1 were separated by strips of wood, as shown in Fig. 3. This was again tested

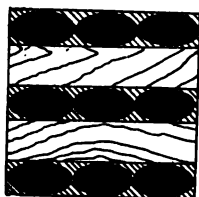


Fig. 3.

when the equatorial planes were normal and when parallel to the field and the results are shown in curves 4 and 3 respectively. Curve 3 shows that the induction, parallel to the major axes, was the same as before the separation occurred, while 4 shows that a separation of the spheroids has decreased the magnetic flux in the direction in which the separation took place. This is an important result.

The spheroids were next set up so that their equatorial planes would make angles other than 0° and 90° with the imposed field. They were tested for 30° , 45° and 60° and the values fell between those of curves 2 and 3. The results seem to show that if one start with equatorial planes parallel to the imposed field that the magnetic flux decreases with the decrease of the diameter of the spheroids parallel to the field. The results of the experiments show that the magnetic flux in a group of spheroids may be influenced in two distinct ways: (1) by the orientation

of the spheroids and (2) by their separation. The flux is greater the more nearly the greatest length is parallel to the field and is decreased by increasing the distance between the surfaces of the spheroids.

DISCUSSION OF RESULTS.

These results obtained by Miss Anderegg throw light on several magnetic phenomena with which we are familiar.

1. *Magnetization Curves.*—In the magnetization curves of soft annealed iron and glass-hardened steel there is a very marked difference in the change of B with H in the two specimens. The knee of the curve occurs at smaller field strengths for soft iron than it does for the hardened steel. Hardened steel makes the best permanent magnets. This would indicate that the elementary magnets are not as free to turn in the hardened steel as in the soft iron. Consequently it will take a greater field strength to orient the elementary magnets a given amount in hardened steel than it does in soft iron. If a part of the increase of B therefore is due to the orientation of the elementary magnets, it must follow that the knee of the induction curve will not occur at as low field strengths for hardened steel as for soft iron, because it will take a larger magnetic field to bring the elementary magnets, which are producing the changes in length, so that their greatest length is parallel to the imposed field and it is at this point that the knee of the induction curve occurs. Previous work has shown that the maximum elongation of hardened steel in the Joule effect occurs at higher values of field strengths than it does in soft iron. If now the change in length may be ascribed to the orientation of the elementary magnets it would seem that we had here a most remarkable relation of phenomena.

2. *Application to the Villari's Reversal Effect.*—As is well known, certain steels when stretched in a weak magnetic field increase their magnetization but if stretched in a strong field lose in magnetization. In a weak magnetic field it is assumed that the elementary magnets are turned more or less in all directions with a tendency for those producing changes in length to turn with equatorial planes parallel to the field imposed upon them. This is shown in the change in length phenomenon. If a pull be applied to a steel rod in a weak field it helps to set the elementary magnets more nearly parallel to the field and so the magnetization is increased. On the other hand if a strong field is applied, *all* of the elementary magnets are turned with the equatorial planes normal to the field. Let a pull be applied to the rod in this last state when the magnetic

¹ Hadley, *Mag. and Elec. for Students*, p. 390.

² Villari, *Pogg. Ann.*, 1868.

field has sufficient power to hold the elementary magnets fixed and the only thing that can occur is a separation of the elements. These experiments show that separating the spheroids decreases the magnetization. This is what occurs when a steel rod is stretched in a strong magnetic field.

3. *Application to Maurain's¹ Experiments.*—Maurain has pointed out that iron electrolytically deposited in a magnetic field shows anisotropic properties, *i. e.*, the elementary magnets seem to have been deposited with a definite orientation and therefore the intensity of magnetization has different values in different directions for the same field strength. Gans² has recently thrown doubt on the anisotropic property of iron but as Vallauri³ has pointed out, Gans worked with too great a field strength (1,250 gauss). At such field strengths the elementary magnets are all turned in a definite direction and no matter in what direction such a field is applied to a specimen of iron it will force all of the elementary magnets with equatorial planes normal to the field. The group of spheroids used in these experiments is a model of the electrolytically deposited iron which shows different intensities of magnetization in different directions and the æolotropic property⁴ of iron is a phenomenon which we should expect to find at proper field strengths.

In crystalline media such as pyrrhotine⁵ we have a very pronounced case of a definite orientation of the elements in which we have different magnetic properties in different directions. It is only a step farther to assume that in the case of crystals showing rotation of the plane of polarization we have another manifestation of this same definite orientation of the elementary particles. This is being investigated in a series of crystals here in our laboratory.

SUMMARY.

1. The results of this investigation show that the orientation of the ellipsoidal elements does affect the magnetic flux. This furnishes a possible explanation as to why the maximum elongation and the knee of the induction curve come at about the same field strength.

2. The behavior of the ellipsoidal elements shows that not only orientation but also the distance between the bounding surfaces of the ellipsoids affect the magnetic properties.

3. This study also points out a possible explanation of the æolotropic properties of iron electrolytically deposited and why in other specimens

¹ Maurain, *Phys. Zeitschr.*, 13, p. 314, 1912. Beetz, *Pogg. Annal.*, CXL., p. 107, 1860.

² Gans, *Phys. Zeitschr.*, 12, p. 911, 1911.

³ Vallauri, *Phys. Zeitschr.*, 13, p. 314, 1912.

⁴ Klemencic, *Phil. Mag.*, p. 424, Vol. 38, 1894.

⁵ Weiss, *Jour. de Phys.*, IV., pp. 469, 829.

of iron there appears to be no æolotropic properties at high field strengths.

4. I have attempted to show that because we have mechanical effects due to magnetization, such as we find in the magnetostrictive effects we must have some hypothesis of magnetization which will account for such phenomena. The ellipsoidal form of the elements, it seems to me, furnishes this part, as the elements are oriented under the influences of the forces operative upon them and produce changes in dimensions.

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March, 1912.

THE MAGNETIZATION OF HEUSLER ALLOYS AS A FUNCTION OF THE TEMPERATURE AND CALCULATION OF THE INTRINSIC MAGNETIC FIELD.

BY PERCY WILCOX GUMAER.

THE magnetic alloys of manganese are composed of metals which ordinarily are non-magnetic. Manganese itself is not only non-magnetic, but a small per cent of it will reduce the magnetic properties of iron. It is probable that the explanation of these magnetic alloys will add considerable to our understanding of the ultimate nature of magnetism.

Recent developments in the electron theory of magnetism have opened up a means of studying the molecular structure of the alloys.

The present investigation was undertaken with two objects in view, first: to study the effect of temperature upon the saturation value of the intensity of magnetization. Then, to determine, if possible, from the data obtained, the structure of the molecular magnets.

METHOD.

To determine the saturation value of the intensity of magnetization the method used by Weiss¹ and Stifler² was chosen. A ballistic galvanometer was connected in series with a helix placed in a strong magnetic field. An ellipsoid of the alloy to be tested was placed in the center of this helix and the deflection of the galvanometer was observed as the ellipsoid was, quickly, pushed out. The deflection of the galvanometer was then compared to that obtained from the current induced in the secondary of a standard helix which was included in the circuit. The intensity of magnetization I can be determined from the relation:

$$I = \frac{k i'}{v d'} d,$$

where k = a constant depending upon the dimensions of the helix and of the standard helix.

v = the volume of the ellipsoid.

i = current in the primary of the standard helix.

d' = deflection due to induced current in the standard helix.

d = deflection when ellipsoid is removed from the helix.

¹ Archives des Sciences, ser. 4, 29, pp. 204 (1910).

² Phys. Rev., Vol. 33, p. 268 (1911).

The ellipsoid and helix were surrounded by a coil of German silver wire, by means of which the desired temperature was obtained. To measure the temperature of the ellipsoid inside the helix a copper-constantan thermo-couple was used. The hot junction was placed in a hard glass tube 3 mm. in diameter and 26 cm. long. Inside the tube the wires were separated by mica strips, outside by 1/16 rubber tubing.

The thermo-couple was calibrated by observing the E.M.F. of the couple when the hot junction was at a known temperature. The temperature of steam and the freezing points of metals were used for the calibration, as follows: Zn 419.4° C., Cd 321.0° C., Sn 231.9° C., steam 100° C. Using the method of least squares the constants of the equation $E = at + bt^2 + ct^3$ were determined and the equation becomes

$$E = 3.747t + .00375t^2 + .00000164t^3.$$

The galvanometer used was a Leeds & Northrup silver suspension instrument. It had a resistance of 25.6 ohms, a ballistic sensibility of 31.8 mm. per micro-coulomb on open circuit, with a scale distance of 50 cm., and a period of 11.2 seconds on open circuit. At a scale distance of 6 meters and the deflection could be read to 0.5 mm.

The ellipsoid was inserted directly into the tube forming the core of the helix. It was moved along by pushing with a small glass rod in one end and with the tube containing the thermo-couple in the other end. By this method the diameter of the helix could be decreased by half, which increased the sensitiveness considerably.

The induction helix was wound upon a thin-walled glass tube 45 cm. long and 0.5 cm. outside diameter. Three layers of number 36 silk-covered copper wire were used. The layers were separated by mica, and each layer was covered with a mixture of water glass and calcined magnesia. This mixture became very hard when dry and held the wires firmly in place even at high temperatures.

A hard glass tube long enough to reach to the end of the bore in the magnet was slipped over the helix coil. Thus the possibility of leakage from the heating circuit to the helix coil or lead-in wires was avoided. The heating coil consisted of one layer of 320 turns of number 16 German silver black enameled wire. The winding was done from the middle towards the ends so that the two halves were wound in opposite directions and opposed each other magnetically. As in the induction helix, a mixture of water glass and magnesia was used to hold the wires firmly in place. Since the length of the coil was 30 cm. the temperature gradient in the center was very small.

The induction helix and heating coil were enclosed in a glass tube small

enough to slide into the bore of the magnet. This tube was filled with calcined magnesia. Further heat insulation between the pole pieces was obtained by enclosing that part of the furnace in a fire clay cylinder filled with shredded asbestos.

The magnetic field was obtained from a large DuBois electromagnet. A hole drilled through the core and pole pieces enabled the ellipsoid to be inserted into the helix. For the air gap used (6.2 cm.) the strength of the magnetic field in the center of the gap was calibrated in terms of the current in the coils. A magnetic balance was used to measure the strength of the field.

DESCRIPTION OF SPECIMENS.

The alloys were prepared by melting in a new graphite crucible heated in a gas furnace. The manganese and copper were put in first, and when they were thoroughly fused the aluminum was added. To insure a uniform mixture, the molten alloy was stirred with a graphite rod, and then quickly poured into vertical moulds. Care was taken to pour in a continuous stream so that the oxide formed on the surface would not injure the casting.

The ellipsoids were obtained by grinding the castings with a properly shaped alundum wheel in a Universal Grinder. A projection of the shadow of the ellipsoids showed the cross-section to be fairly accurate.

The dimensions and composition of the two ellipsoids are given as follows:

	Ellipsoid No. 1.	Ellipsoid No. 2.
Length.....	1.686 cm.	1.680 cm.
Mean diameter.....	0.393 cm.	0.399 cm.
Volume.....	0.1364 cu. cm.	0.1401 cu. cm.
Mass.....	0.9487 gm.	1.0028 gm.
Density.....	6.96	7.15
Copper.....	62.9 per cent.	61.95 per cent.
Manganese.....	18.5 per cent.	21.9 per cent.
Aluminum.....	15.1 per cent.	15.9 per cent.
Undetermined.....	3.5 per cent.	0.25 per cent.

PROCEDURE IN TAKING READINGS.

After the heating current had been on for a time, sufficient to establish temperature equilibrium, the ellipsoid was inserted into the core of the helix. It was moved along by pushing with the tube containing the thermo-couple from one end and with a glass rod from the other end. A mark on the glass rod indicated when the ellipsoid was in the

center of the helix. When the temperature had ceased to increase the reading of the thermo-couple was taken, the magnetic field was thrown on and the deflection of the galvanometer was observed as the ellipsoid was quickly pushed out of the helix. A rapid movement of the ellipsoid was obtained by striking the end of the glass rod with a small piece of wood.

The ellipsoid was now replaced in position, allowed to regain its former temperature and the reading repeated. As a rule three readings were taken for each field strength and the intensity of magnetization was calculated from a mean of the three deflections.

At lower temperatures the deflections agreed to within 1 per cent. but in the neighborhood of the transformation temperature the agreement was not as close. For some of the readings taken above 300° the maximum deflection was 1 cm. at a scale distance of 6 meters. The accuracy in this case was probably about 10 per cent.

After each set of readings the galvanometer was calibrated by means of the standard helix. The current in the primary of the helix was read by a Weston milli-ammeter, which had been calibrated by comparison with a standard instrument. The ratio i'/d' varied slightly as the temperature increased, due to the increased resistance of the helix coil at higher temperatures. As the heating coil was wound non-magnetically, it was not necessary to make any correction for it.

Thermo-couple readings were taken just before the ellipsoid was pushed out of the helix. As the end of the tube containing the thermo-couple was left open, and as the couple was

within a millimeter of the end of the ellipsoid when readings were taken, it is quite probable that the temperature measured corresponded very accurately to the actual temperature of the ellipsoid.

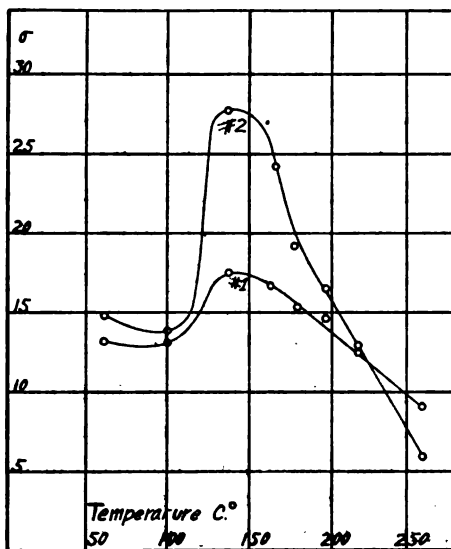


Fig. 1.

RESULTS.

The first set of data obtained is apparently of little theoretical value. The curves (Fig. 1) showing the specific intensity of magnetism σ as a

function of the temperature are quite irregular, having a maximum at 140° . Although it is possible that the irregularity of these curves is due to a defect in the apparatus, it is more probable that it is due to the unstable conditions of the alloys. The data were obtained with the alloys in the condition as cast and without previous heat treatment.

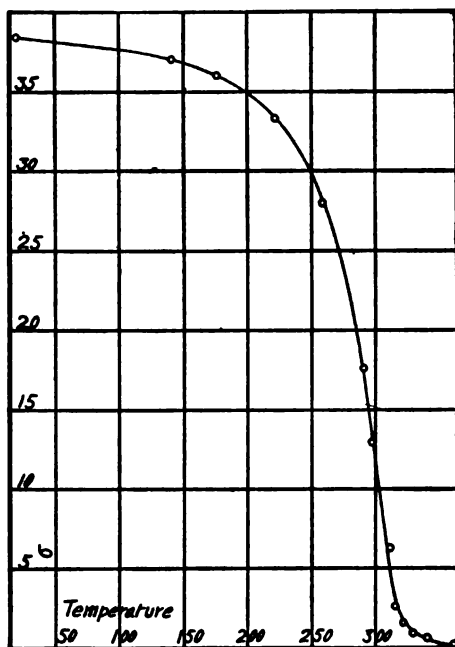


Fig. 2.
Ellipsoid No. 1.

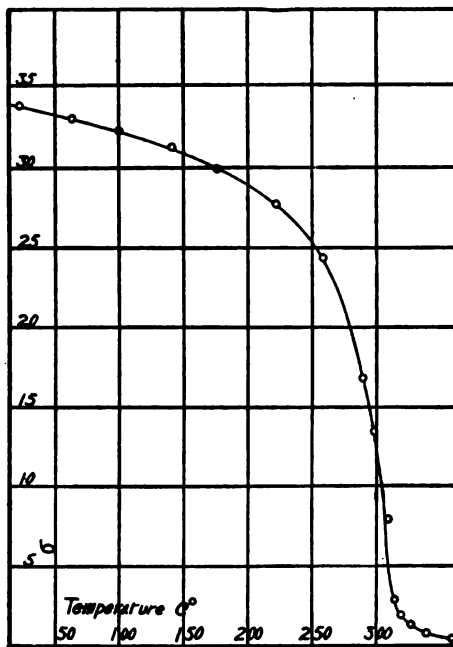


Fig. 3.
Ellipsoid No. 2.

A new helix coil was now built and a series of readings taken at 320° indicated that the substance had become paramagnetic. Beginning at room temperature, the whole set of data was repeated and very regular curves were obtained, as shown in Figs. 2 and 3. These curves, showing σ , the specific intensity of magnetization, as a function of the temperature, are similar to the ones obtained for iron, nickel and cobalt, although they are much flatter at lower temperatures. It is seen from the curves that the temperature of transformation is in the neighborhood of 310° . A theoretical discussion of these curves will be given in a later paragraph.

As the values of σ at room temperature were found to be about half of what should be expected from the theoretical calculations, an attempt was made to increase the magnetic intensity by chilling from a temperature near the melting point of the alloy. To do this the ellipsoids were inserted in a quartz tube together with a platinum platinum-rhodium

thermo-couple, and heated in an electric furnace to 895°C . They were kept at the temperature for 10 minutes and then chilled by plunging the quartz tube into cold water.

The value of the intensity of magnetization was found to have been increased considerably by the chilling. Values of σ were obtained, as before, for different values of temperature below the transformation point. The curves are shown in Fig. 4.

In order to be sure of the results the chilling was repeated for ellipsoid No. 1 and similar values were obtained.

The following table shows a typical set of readings taken at 290°C . The value of σ at each field strength was obtained from the mean of three galvanometer deflections. The temperature was obtained from the mean of the thermo-couple readings. These were not allowed to vary more than 20 micro-volts, which corresponds to 0.4° .

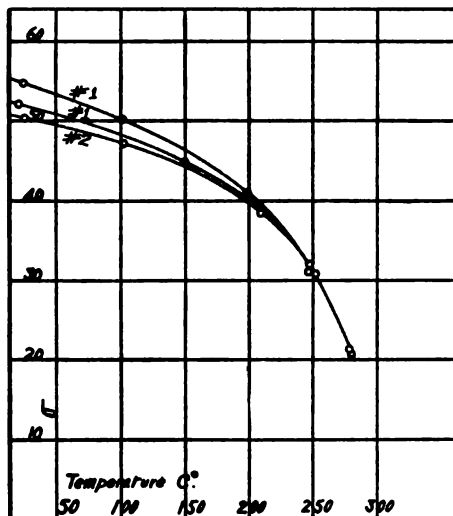


Fig. 4.

m	d	Mean d	σ	I	$L.I$	I_m	H_0	H
13,670	27.5	27.4	15.9	114.0	85	2.0	930	845
13,680	27.3							
13,675	27.5							
13,670	27.8	27.7	16.1	115.3	86	3.0	1,395	1,309
13,680	27.7							
13,670	27.7							
13,675	29.0	29.0	16.8	120.5	90	4.0	1,860	1,750
13,665	28.8							
13,665	29.2							
13,670	29.0	28.9	16.75	120.0	90	5.0	2,335	2,245
13,670	28.7							
13,675	29.0							
13,670	29.0	28.9	16.75	120.0	90	6.0	2,730	2,640
13,670	28.8							
13,665	28.9							

TYPICAL SET OF DATA. TAKEN AT 290.9°.

Ellipsoid No. 2.

$$I = 4.157d, \quad \sigma = 0.580d, \quad L = 0.749.$$

d = deflection of the galvanometer,

mv = reading of thermo-couple in microvolts,

I_m = current through the magnet,

H_0 = external field,

H = field inside the ellipsoid = $H_0 - LI$,

I = intensity of magnetization,

σ = specific intensity of magnetization.

Mean value of thermo-couple readings = 13,670 m.v.

Corresponding temperature = 290.9°.

Deflection due to standard helix = 70.3.

Current in primary of standard helix = 0.672 ampere.

MOLECULAR THEORY OF MAGNETISM.

The present theory of magnetism, as developed by Curie,¹ Weiss,² Langevin³ and Kunz,⁴ accounts for the various phenomena by assuming that a magnetic substance is made of small elementary magnets.

In a non-magnetic state these elementary magnets are distributed with their axes pointing equally in all directions. Under the influence of a resultant magnetic field H , each elementary magnet is acted upon by a turning force $MH \sin \alpha$, where M is the moment of the elementary magnet and α is the angle between H and the axis of the magnet. The tendency of this couple is to cause the magnets to turn with their axes toward the direction of the existing field. The amount of this rotation depends upon the strength of the field and upon the temperature of the substance.

If there were no thermal agitation of the molecules all the elementary magnets would revolve until their axes coincided with the direction of the existing field. This condition is obtained at absolute zero.

At other temperatures than absolute zero the magnetic energy of the molecules tending to arrange the molecules in the direction of the magnetic field is opposed by the thermal energy. The molecules are continually being deflected by their mutual collisions, and the resultant condition of equilibrium depends upon the ratio of the thermal energy to the magnetic energy.

¹ Archives des Sciences, ser. 4, 31, p. 5-19 (1911).

² Journal de Physique, 36, p. 661-690 (1907).

³ Annales de Chemie et de Physique, Ser. 5, 8, p. 70-127 (1905).

⁴ Phys. Rev., 30, p. 359-370 (1910).

Consider a sphere of unit radius within which are a large number of magnetic molecules. When there is no magnetic field acting, the magnets are distributed with their axes pointing equally in all directions. Imagine all these magnets concentrated with their centers at o . Under the action of a magnetic field the magnets will be caused to rotate about o , and the tendency will be to place their axes in line with the magnetic field. The magnets will no longer have their axes pointing uniformly in all directions, but the magnetic density will be greatest in the direction of H . Let us define magnetic density as the number of magnetic axes per unit solid angle or $dn/d\omega$. For abbreviation put $\rho = dn/d\omega$.

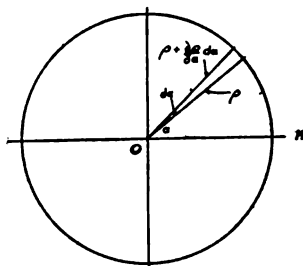


Fig. 5.

Let ρ be the magnetic density at any angle α with the field H . Then at angle $\alpha + d\alpha$ the magnetic density will be $\rho + (\partial\rho/\partial\alpha)d\alpha$. The change of magnetic density in moving through angle $d\alpha$ is therefore

$$\rho - \left(\rho + \frac{\partial\rho}{\partial\alpha} d\alpha \right) = - \frac{\partial\rho}{\partial\alpha} d\alpha.$$

This change of density depends upon the density ρ at α . It is also proportional to a resultant turning force or couple.

If we assume that the molecules of iron, or other ferro-magnetic substance, when in a non-magnetic state are as free to move relatively to each other as the molecules of a gas, then the thermal energy can be deduced from the laws of thermo-dynamics. Since a rotation of the elementary magnet about its own axis has no effect upon its magnetic energy, there remain but two degrees of rotation. Hence, for the molecular magnets, the kinetic energy of heat is equal to RT , where R is the universal gas constant and T is the absolute temperature.

In dynamics work/angle = a couple, hence $RT/d\alpha$ = a couple due to the thermal energy of the molecules. The magnetic couple acting upon the elementary magnets is $MH \sin \alpha$.

The change of magnetic density varies directly as the magnetic couple and inversely as the couple due to the heat energy; that is, the greater the magnetic couple, the greater will be the change of magnetic density as we pass from α to $\alpha + d\alpha$, and the greater the thermal energy, the smaller will be the change of density. Hence we can put

$$- \frac{\partial\rho}{\partial\alpha} d\alpha = \rho \frac{MH \sin \alpha}{\frac{RT}{d\alpha}} = \rho \frac{MH \sin \alpha \cdot d\alpha}{RT}.$$

Then

$$-\frac{\partial \rho}{\partial \alpha} = \rho \frac{MH}{RT} \sin \alpha;$$

and

$$-\frac{\partial \rho}{\rho} = \frac{MH}{RT} \sin \alpha d\alpha.$$

Integrating

$$\log \rho = \frac{MH}{RT} \cos \alpha - \log K;$$

whence,

$$\rho = Ke^{\frac{MH}{RT} \cos \alpha},$$

or

$$\frac{dn}{d\omega} = Ke^{\frac{MH}{RT} \cos \alpha}.$$

But

$$d\omega = 2\pi \sin \alpha d\alpha;$$

hence

$$dn = Ke^{\frac{MH}{RT} \cos \alpha} 2\pi \sin \alpha d\alpha. \quad (1)$$

Intensity of magnetization may be defined as the product of the number of molecular magnets per unit volume and the moment of the magnets in the direction of the resultant magnetic field. The intensity is a maximum when there is no thermal agitation, so that the molecular magnets are all directed along the field. This condition obtains only at absolute zero.

At other temperatures the intensity due to the magnets whose axes make an angle α with H is

$$dI = M \cos \alpha dn,$$

where M is the moment of the molecular magnets, and I is the intensity of magnetization.

Substituting the value of dn from eq. (1), we have

$$dI = M \cos \alpha Ke^{\frac{MH}{RT} \cos \alpha} 2\pi \sin \alpha d\alpha;$$

and integrating between limits 0 and π

$$I = \int_0^\pi M \cos \alpha Ke^{\frac{MH}{RT} \cos \alpha} 2\pi \sin \alpha d\alpha.$$

Now let

$$\frac{MH}{RT} = a, \quad \cos \alpha = x, \quad -\sin \alpha d\alpha = dx;$$

then,

$$I = 2\pi MK \int_{+1}^{-1} x e^{ax} dx.$$

But

$$\begin{aligned} \int_{+1}^{-1} x e^{ax} dx &= \frac{a(e^e + e^{-e})}{a^2} - \frac{(e^e - e^{-e})}{a^2} \\ &= 2 \left(\frac{\cosh a}{a} - \frac{\sinh a}{a^2} \right); \end{aligned}$$

hence

$$I = 2\pi MK_2 \left(\frac{\cosh a}{a} - \frac{\sinh a}{a^2} \right). \quad (2)$$

To evaluate K , integrate equation (1):

$$\begin{aligned} n &= 2\pi K \int_0^\pi e^{a \cos \alpha} \sin \alpha d\alpha \\ &= \frac{4\pi K}{a} \sinh a; \end{aligned}$$

whence

$$K = \frac{na}{4\pi \sinh a}.$$

Substituting the value of K in equation (2), we have

$$\begin{aligned} I &= 2.2\pi M \frac{na}{4\pi \sinh a} \left(\frac{\cosh a}{a} - \frac{\sinh a}{a^2} \right) \\ &= Mn \left(\frac{\cosh a}{\sinh a} - \frac{1}{a} \right). \end{aligned}$$

But $Mn = I_m$; hence,

$$I = I_m \left(\frac{\cosh a}{\sinh a} - \frac{1}{a} \right), \quad (3)$$

where

$$a = \frac{MH}{RT}. \quad (4)$$

Equations (3) and (4) give us an expression for the intensity of magnetization as a function of the temperature.

So far we have considered only the arrangement of the elementary magnets due to the action of an external field. Each magnet, however, has an effect upon the surrounding magnets and the result according to the Weiss theory is a uniform field, proportional to the intensity of magnetization, I , and acting in the same direction as I . This molecular magnetic field accounts for the great magnetic intensity of iron and

other ferro-magnetic substances in the same way as an internal pressure added to the external pressure accounts for the great density of liquids. The sudden increase of density when a vapor liquefies is due to the fact that an enormous internal pressure is suddenly made effective in addition to the external pressure. Ferro-magnetic substances at high temperatures are but slightly magnetic. As the temperature is slowly decreased a point is reached at which the substances suddenly become very magnetic. This indicates that a strong molecular field has become operative. If H_m represents the molecular field and I the intensity of magnetization then $H_m = AI$ where A^1 is a proportionality factor.

The resultant magnetic field within the substance is, consequently, the sum of the external field, H_e , and the molecular field H_m , or, $H = H_e + H_m$.

From equation (4)

$$T = \frac{MH}{aR};$$

whence

$$T = \frac{M(H_e + H_m)}{aR}. \quad (5)$$

The large magnetic intensity of ferro-magnetic substances at ordinary temperatures indicates that the molecular field must be very strong in comparison with the external field. This condition holds up to the temperature at which the substance ceases to be ferro-magnetic. Let θ be that temperature, then for $T = \theta$, we can neglect H_e in comparison to H_m and equation (5) becomes

$$T = \frac{MAI}{aR}, \quad \text{for } T \leq \theta. \quad (5a)$$

From equation (3)

$$I = I_m \left(\frac{\cosh a}{\sinh a} - \frac{1}{a} \right).$$

The expression $\left(\frac{\cosh a}{\sinh a} - \frac{1}{a} \right)$ can be expanded into the convergent series

$$\frac{1}{3}a - \frac{2}{90}a^3 + \frac{4}{45 \cdot 42}a^5 \dots$$

For very small values of a it is sufficient to consider only the first term of the series, and equation (3) becomes

$$I = \frac{aI_m}{3}. \quad (6)$$

¹ In order to avoid ambiguity later the symbol A is chosen in preference to N , the symbol used by Weiss, Kunz and others.

For $T = \theta$ equation (5) may be written

$$\theta = \frac{MAI_m}{3R}. \quad (7)$$

Dividing equation (5) by (7), we have

$$\frac{T}{\theta} = \frac{\frac{M(H_0 + AI)}{aR}}{\frac{MAI_m}{3R}} = \frac{3H_0}{aAI_m} + \frac{3I}{aI_m}.$$

But

$$I = \frac{a}{3} I_m;$$

hence

$$\frac{T}{\theta} = \frac{H_0}{AI} + 1.$$

Solving for A , we have

$$A = \frac{H_0}{I} \frac{\theta}{T - \theta}. \quad (8)$$

Knowing A , we can calculate H_m , the strength of the molecular field, since $H_m = AI$.

We can also calculate M , the moment of the elementary magnet, for from equation (7) we have

$$\theta = \frac{MAI_m}{3R},$$

or

$$M = \frac{3R\theta}{AI_m}. \quad (9)$$

DISCUSSION OF RESULTS.

The curves (2) and (3), showing σ as a function of the temperature indicate that the temperatures at which the alloys cease to be ferromagnetic is in the neighborhood of 310°C .

Knowing θ it is possible to express T as a function of the parameter a . Then by comparing the graph obtained with the experimental curve near the transformation point, we can calculate a value for I_m , the intensity of magnetization at absolute zero.

Equation (5a) can be written in the form

$$I = \frac{aRT}{MA}.$$

Dividing through by I_m , we have

$$\frac{I}{I_m} = \frac{R}{MAI_m} aT.$$

But from equation (7)

$$\frac{R}{MAI_m} = \frac{1}{3\theta};$$

hence

$$\frac{I}{I_m} = \frac{1}{3\theta} aT,$$

and

$$T = \frac{3\theta}{a} \frac{I}{I_m} = \frac{3\theta}{a} \left(\frac{\cosh a}{\sinh a} - \frac{1}{a} \right). \quad (12)$$

Putting $\theta = 310^\circ + 273^\circ = 583^\circ$ and evaluating equation (12), we obtain the following values:

a	$\frac{\cosh a}{\sinh a}$	$\frac{\sigma}{\sigma_m} = \frac{\cosh a}{\sinh a} - \frac{1}{a}$	T	t	$74.5 \frac{\sigma}{\sigma_m}$
0.3	3.4328	0.0995	580	307	7.0
0.4	2.6317	0.1317	577	304	9.8
0.5	2.1639	0.1639	574	301	12.2
0.6	1.8619	0.1952	569	296	14.5
0.8	1.5059	0.2559	560	287	19.1
1.0	1.3131	0.3131	548	275	23.3
1.2	1.1995	0.3662	534	261	27.3
1.6	1.0849	0.4599	503	230	34.3
2.0	1.0373	0.5373	470	197	39.9
3.0	1.0049	0.6716	392	119	50.0
4.0	1.0007	0.7507	328	55	55.9
5.0	1.0000	0.8000	280	7	59.6

Since

$$\frac{I}{I_m} = \frac{\sigma}{\sigma_m},$$

$$\sigma_m = \frac{\sigma}{\frac{\cosh a}{\sinh a} - \frac{1}{a}},$$

where σ_m is the value of σ at absolute zero.

In the following table the experimental values of σ are compared with the calculated values of σ/σ_m for values of t above 250° .

t	σ Observed.	σ/σ_m Calculated.	Value of σ_m .
297.7	12.9	.184	70.1
297.7	13.5	.184	73.4
290.9	17.6	.234	75.5
290.9	16.8	.234	71.8
258.9	28.5	.372	76.6
			74.5 mean

It is evident that if we divide the observed values of σ by the calculated values of σ/σ_m we can determine the value of σ_m . Taking the mean of several values, we get $\sigma_m = 74.5$. If we now multiply the values of $\frac{\cosh a}{\sinh a} - \frac{1}{a}$ by 74.5, we obtain corresponding values for σ and t . Using these values we can plot a theoretical curve for σ and the temperature. Such a curve is shown in Fig. 6.

It will be noticed that the experimental values of σ as given by the dotted lines agree with the theoretical values for high temperatures, but at low temperatures the experimental curve bends away from the other. The same phenomenon is observed in iron, nickel and cobalt, but to a lesser degree. In the case of the magnetic alloys the bending is less (curve B) after the alloy has been chilled from a high temperature.

Since the internal or molecular field becomes negligible above the transformation temperature, the intensity of magnetization becomes proportional to the external field, and from a knowledge of the magnetic properties of the alloy in its paramagnetic state we calculate H_m the intrinsic molecular field and m the moment of the elementary magnets.

In doing this we will first calculate A , the proportionality constant.

From equation (8) we have

$$A = \frac{H_e}{I} \frac{\theta}{T - \theta},$$

where H_e is the external field and I is the intensity of magnetization at temperature T .

Calculations of A are given in the following table for various values of T . They give for ellipsoid No. 1 $A = 12,940$; for ellipsoid No. 2 $A = 10,540$.

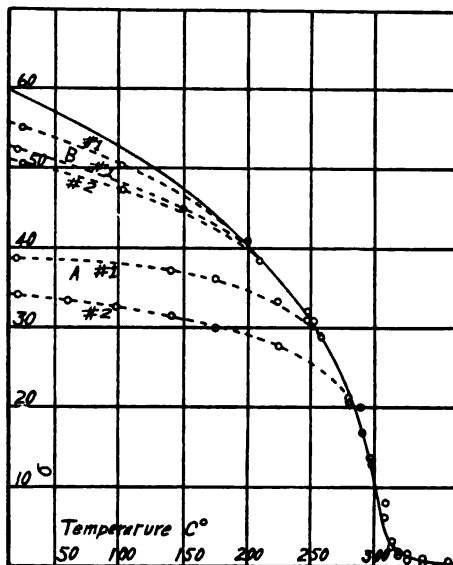


Fig. 6.

Ellipsoid No. 1.

Temp. = 314.5°.			Temp. = 320°.		
H_e	I	$K = \eta H_e$	H_e	I	K
462	3.93	.00851	465	1.98	.00427
924	7.42	.00803	927	4.02	.00434
1,382	14.4	.01041	1,390	6.23	.00448
1,844	17.9	.00973	1,853	7.56	.00408
2,319	18.2	.00788	2,326	10.25	.00440
Mean .00891		$A = 14,560$	Mean .00439		$A = 13,270$
Temp. = 327.7°.			Temp. = 340°.		
H_e	I	K	H_e	I	K
464	1.31	.00282	1,394	1.79	.001284
928	2.62	.00282	2,726	4.15	.001521
1,392	3.92	.00282	3,425	5.68	.001658
1,855	5.23	.00282	3,695	7.0	.001894
2,329	6.54	.00281			
Mean .00282		$A = 11,700$	Mean .00159		$A = 12,230$

Ellipsoid No. 2.

Temp. = 314.5°.			Temp. = 320°.		
H_e	I	$K = \eta H_e$	H_e	I	K
463	4.61	.00996	463	2.26	.00489
923	9.66	.01046	926	5.83	.00630
1,384	15.14	.01094	1,389	7.96	.00573
1,846	18.92	.01026	1,852	10.83	.00584
			2,326	12.00	.00518
Mean .01040		$A = 12,460$	Mean .00559		$A = 10,430$
Temp. = 340°.			Temp. = 360°.		
H_e	I	K	H_e	I	K
1,393	2.74	.001966	1,394	1.904	.001365
2,726	5.77	.002115	2,728	3.38	.001240
3,425	6.74	.001967	3,427	4.23	.001233
3,825	7.37	.001926	3,827	4.02	.001050
Mean .001993		$A = 9,760$	Mean .001227		$A = 9,500$

Ellipsoid No. 1.		Ellipsoid No. 2.	
Temp.	A	Temp.	A
314.5	14,560	314.5	12,460
320.0	13,270	320.0	10,430
327.7	11,700	340.0	9,760
340.0	12,230	360.0	9,500
Mean $A = 12,940$		Mean $A = 10,540$	

The calculation of the other quantities consists merely in substituting in the equations already obtained. The work is summarized as follows:

	Ellipsoid No. 1.	Ellipsoid No. 2.
σ_m ,	74.5	74.5
d = density,	6.96	7.15
$I_m = d\sigma_m$,	518	533
$A = \frac{He}{I} \frac{\theta}{T - \theta}$, (8)	12,940	10,540
$H_m = AI_m$,	6,700,000	5,620,000
$M = \frac{3R^1\theta}{AI_m}$, (9)	$3.55 \cdot 10^{-20}$	$4.23 \cdot 10^{-20}$
$N = \frac{I_m}{M}$. (10)	$1.46 \cdot 10^{23}$	$1.26 \cdot 10^{23}$

$$R = 1.36 \cdot 10^{-16}.$$

For comparison, the constants obtained for iron and nickel by Kunz² and for cobalt by Stifler³ are given in the following table:

	I_m	θ	A	H_m	$M \times 10^{-20}$.	$N \times 10^{23}$.
Iron	2,120	756° C.	3,850	6,560,000	5.15	4.12
Cobalt	1,435	1,075	6,180	8,870,000	6.21	2.31
Nickel	570	376	12,700	6,350,000	3.65	1.56
Alloy No. 1.	518	310	12,940	6,700,000	3.55	1.46
Alloy No. 2.	533	310	10,540	5,620,000	4.23	1.26

If we assume that there is one atom of manganese in the magnetic molecule then for alloy number one we have the relation

$$m_H = \frac{d}{wn},$$

where m_H = weight of a hydrogen atom.

$w = 55$ = atomic weight of manganese.

$n = 1.46 \cdot 10^{23}$ the number of magnetic molecules per cu. cm.

d = density of alloy $\times$ per cent. of manganese in alloy.

$$= 6.96 \cdot 0.185.$$

$$m_H = \frac{6.96 \cdot 0.185}{55 \cdot 1.46} 10^{23} = 1.60 \cdot 10^{-24}.$$

¹ R is the universal gas constant, and the value to be used is that corresponding to one molecule.

² PHYS. REV., Vol. 30, p. 259 (1910).

³ PHYS. REV., Vol. 33, p. 268 (1911).

This value of m_H agrees almost exactly with $1.61 \cdot 10^{-24}$, the value obtained by Rutherford. From alloy number two we get $m_H = 2.12 \cdot 10^{-24}$, which does not agree very closely.

If e is the elementary charge of the hydrogen atom and α_H is the chemical equivalent of hydrogen, then $e = m_H/\alpha_H$. Using the values of m_H obtained above we get from alloy number one $e = 1.54 \cdot 10^{-20}$ and from alloy number two $e = 2.04 \cdot 10^{-20}$.

We could have obtained, however, the same values by assuming that the magnetic molecule was composed of one atom of manganese and one atom of aluminium, or one atom of manganese and one atom of copper. Since a molecule must contain more than one atom, it is quite probable that the magnetic molecule is a composite molecule containing one atom of manganese and one atom either of copper or of aluminium. This hypothesis would also account for the increase in the intensity of magnetism after chilling from a high temperature, as shown in Fig. 6. The chilling prevents one of the metals from crystallizing out and thus prevents a decrease in the number of molecular magnets.

The author hopes to continue the investigation and to determine definitely the number and kind of atoms in the elementary magnet, and also to study the effect of the percentage of copper upon the transformation temperature. These results will be important in proving that the magnetic properties are due to the manganese.

SUMMARY.

The chief results of this investigation may be summarized as follows:

1. The temperature of magnetic transformation from the ferro-magnetic to the paramagnetic state was established at 310° C. for the alloys containing 62 per cent. copper.
2. The curve giving σ as a function of the temperature has been shown to agree with the theoretical curve above 200° .
3. Chilling from near the melting point causes the experimental curve to follow the theoretical curve to a lower temperature than before.
4. The nature of the molecular field was found to be of the same order of magnitude as nickel, all the constants I_m , H_m , M and H , being of approximately the same value.
5. The results, while not extensive enough to determine the number and kind of atoms in the elementary magnet, are sufficient to show that the alloys obey the laws of ferro-magnetism, as derived by the present molecular theory.
6. The fundamental equation (1), on which the present theory of

magnetism is based, has been derived mathematically; thereby, making the former analogy to the gas theory unnecessary.

The writer takes pleasure in acknowledging his indebtedness to Professor Jakob Kunz for his general supervision of the work, and for many valuable suggestions.

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UNIVERSITY OF ILLINOIS,
April 25, 1912.

NOTES ON OPTICAL CONSTANTS OF METALS.

BY J. T. LITTLETON, JR.

INTRODUCTION.

THIS work consists in the determination of the values of the optical constants, that is, n the refractive index and k the absorptive index, for silicon and ductile tungsten, and a study of the effect of heat treatments on different steels. Observations have been made on silicon and tungsten by von Wartenberg¹ and on silicon by Ingersoll and Littleton.² The specimens used in this investigation are much purer than the one used by von Wartenberg, and, in addition to observations (by another method), on the same piece studied by Ingersoll and Littleton measurements are made on a piece with a different crystalline structure, and on one showing the Hall and thermo-electric effects in a direction different from the others. The metallic tungsten used in this work is of an entirely different form from the one used by von Wartenberg.

Many observers have measured the optical constants of steel, but with no concordance of results. This lack of agreement among different experimenters must be largely due to the fact that measurements were made on specimens of entirely different carbon content and having no similarity as regards heat treatment.

Hagen and Rubens³ give the values of 55.4 per cent. for the reflecting power of soft steel and of 60.0 per cent. for hard steel for light of a wavelength of $6,000\mu$ and Drude⁴ using sodium light gives for steel the value 58.5 per cent., but in neither case is the carbon content specified or is the heat treatment prescribed other than merely by the words hard and soft. But it is well known that these two factors have a great influence on all of the physical properties of the metal. It is therefore of interest to see whether the optical properties are so affected, and whether by this means the apparent disagreement between different experimenters can be explained.

EXPERIMENTAL METHOD.

Since the method used for the determination of the values of n and k is that used by Drude,⁵ no experimental details need be given.

¹ D. Phys. Gesell. Verh., 12, p. 105, 1910.

² Phys. Rev., 31, Nov., 1910.

³ Zeits. f. Instrumentk., 22, 52, 1902.

⁴ Ann. Phys., 39, 481, 1890.

⁵ Ann. Phys., 39, 481, 1890.

All grinding was done with graded carborundum, and the polishing with rouge on a pitch tool or with rouge on broad-cloth. But since this latter method has a tendency to give a relief polish it is not desirable in all cases.

As a source of light fused sodium chloride in a blast lamp was used for the measurements on silicon and tungsten, but the Meker burner with a platinum holder for the sodium chloride used for the steels proved much more satisfactory.

Observations were made at principal incidence and principal azimuth, that is, at that angle of incidence, at which light plane polarized at an angle of 45° with the plane of incidence, becomes circularly polarized after reflection from the metallic surface. Angles of incidence could be read to $30''$ and azimuth to 0.1° .

If $\bar{\phi}$ be the angle of principal incidence and $\bar{\psi}$ the principal azimuth the well-known Drude formulæ become for this position:

$$n = \sin \bar{\phi} \tan \bar{\phi} \cos 2\bar{\psi} + \frac{\cos \bar{\phi} \cos 2\bar{\psi}}{2},$$

$$k = \tan 2\bar{\psi}(1 - \cot^2 \bar{\phi}).$$

The reflecting power is calculated from n and k by:

$$R = \frac{n^2(1 + k^2) + 1 - 2n}{n^2(1 + k^2) + 1 + 2n}.$$

I. SILICON.

Observations were taken on four pieces of silicon obtained from the Carborundum Co., of Rochester, N. Y., having a purity of 99.75 per cent. No. 1 had a good black polish, though not free from scratches, No. 2 was from the same plate as No. 1, and was scratched in only one direction with carborundum which had remained suspended in water for five minutes, No. 3 was the flat surface of a crystal and No. 4 was a piece giving thermoelectric effects and Hall effects in a direction opposite to the others. Since these give the same values for n and k to within the possible experimental error, results will be given for the first piece only, this having the best polish of the four. This is also the same plate on which measurements out in the infra-red to 2.5μ were made by Ingersoll,¹ and by Ingersoll and Littleton² by another method.

The means of ten observations at angles of incidence equal to 68° , 70° , 72° , 74° , and 76° give:

¹ Astr. Phy. Jour., Vol. XXXII., p. 265, 1910.

² Phys. Rev., Vol. XXXI., p. 489, 1910.

$n = 4.24$, $k = 0.118$, $\bar{\phi} = 76^\circ 45'$, $2\bar{\psi} = 6^\circ 52'$, $R = 37.8$ per cent

The greatest deviation from the mean was 1.5 per cent. for n and 7 per cent. for k , ϕ and 2ψ were calculated and the above value of R is the calculated reflecting power. R measured by a Martens-Koenig spectrophotometer is 37.7 per cent. This result is accurate to 0.5 per cent. The reflecting power for the Rest-strahlen from quartz of 9μ wave-length was measured by means of a Rubens thermopile and was found to be 33.5 per cent.

Von Wartenberg using silicon 95 per cent. pure and light of a wave-length 5.790μ gets:

$n = 3.87$, $k = 0.116$, $\bar{\phi} = 75^\circ 38'$, $2\bar{\psi} = 7^\circ 3'$, $R = 35.7$ per cent. This difference between the two determinations is readily explained by the difference in purity.

II. TUNGSTEN.

A thin plate of ductile tungsten having the dimensions 1.5 by 0.5 square centimeters was obtained from the General Electric Co. Scratches did not have much effect on either principal incidence or principal azimuth, but the final observations were on a surface practically free from scratches.

The means of ten sets of measurements of $\bar{\phi}$ and $2\bar{\psi}$ gave:

$n = 3.46$, $k = 0.94$, $\bar{\phi} = 78^\circ 31'$, $2\bar{\psi} = 44^\circ 3'$, $R = 54.5$ per cent.

Von Wartenberg using a form of pressed tungsten and light of a wave-length equal to 5.790μ gets the following results:

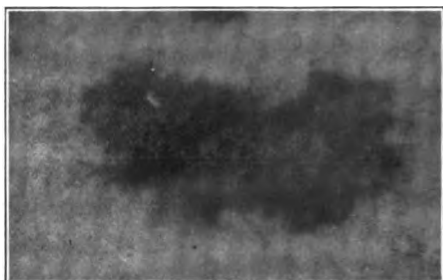
$n = 2.76$, $K = 0.98$, $\bar{\phi} = 76^\circ 0'$, $2\bar{\psi} = 46^\circ 20'$, $R = 48.6$ per cent.

This difference is no doubt due to the different form of metal used.

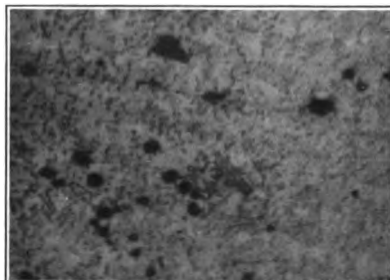
III. THE EFFECT OF HEAT TREATMENTS AND CARBON CONTENT ON THE OPTICAL PROPERTIES OF STEEL.

Measurements were made on five surfaces of 0.44 per cent. carbon steel, on five of 1.28 per cent. carbon steel, on one of about 3.5 per cent. carbon and one of pure iron. These steels were obtained from Prof. E. D. Campbell of the department of chemistry and are of accurately known composition and heat treatment.

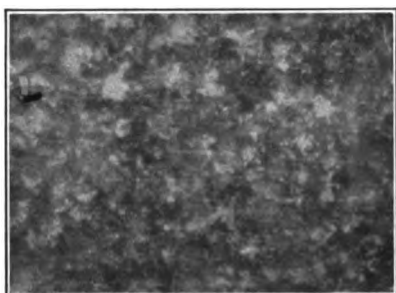
Steels may be divided into four classes, the basis of division being the crystalline structure as controlled by heat treatments. CLASS I. is a mixture of the segregated particles of the carbides of iron in a matrix of pure iron, and is obtained by holding the metal at a temperature of about 600° C. for twenty-four hours or longer, which treatment allows the carbides of iron to collect into large particles. CLASS II. is very similar, differing only in degree, the particles of the iron carbides being very small.



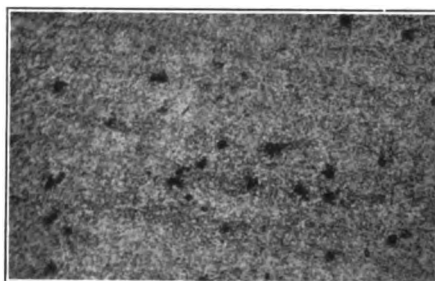
No. 1D.
High power.



No. 1D.
Low power.



No. 5D.
High power.



No. 5D.
Low power.

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This structure is obtained by slow annealing from about 900° C. or by quenching from 900° C. and reheating to about 600° C. and holding at this temperature for eight or ten hours. The carbides begin to go out of solution and to collect into very fine particles. This form of steel is called *perlite*. CLASS III. is, according to Campbell,¹ a supersaturated solution of the carbides of iron in α iron. This structure is obtained by quenching from about 900° C. and reheating to about 340° C. and holding at this temperature for ten or more hours. This structure is called *troostitic*. In opposition to Campbell's view is the theory that this steel consists of very finely divided particles of the carbides of iron mechanically mixed in the iron crystal. CLASS IV. is the quenched or hardened steel and is obtained by holding the specimen at a temperature around 900° C. for thirty minutes and quenching in ice water. This gives a solid solution of the carbides of iron in small γ iron crystals.

These treatments cause quite a difference between the two extremes of structure, that is, between the segregated iron and the quenched and hardened. The micro-photographs of the 1.28 per cent. carbon steel show this characteristic difference. In the segregated iron the segregated particles are clearly shown under the low power magnification and one large particle is shown under the high power. The difference between the two structures is clearly visible. Microscopic examination of the surfaces prepared from the other specimens showed them to be good examples of their class.

After microscopic examination the etching was ground off and the surface repolished and the observations taken. The means of ten observations on these surfaces are given in the following table.

Optical Constants of Steel.

No.	Per Cent. C.	ϕ	2ψ	n	h	R
No. 1C	0.44	77° 10'	53° 55'	2.59	1.30	57.2
No. 2C	0.44	77° 11'	53° 55'	2.59	1.30	57.2
No. 3C	0.44	77° 18'	53° 55'	2.61	1.30	57.5
No. 4C	0.44	77° 16'	53° 55'	2.61	1.30	57.5
No. 5C	0.44	77° 18'	53° 55'	2.61	1.30	57.5
No. 1D	1.28	77° 16'	53° 30'	2.63	1.28	57.4
No. 2D	1.28	77° 26'	53° 30'	2.67	1.28	57.4
No. 3D	1.28	77° 12'	53° 30'	2.63	1.28	57.6
No. 4D	1.28	77° 30'	53° 30'	2.68	1.28	57.5
No. 5D	1.28	77° 28'	53° 30'	2.68	1.28	57.5
No. 1E	3.5	77° 35'	52° 24'	2.77	1.23	57.0
No. 1F	0.00	77° 1'	54° 10'	2.53	1.31	56.2
Drude	Steel	77° 8'	55° 38'	2.41	1.41	58.5
Drude	Iron	76° 30'	54° 58'	2.36	1.36	56.2

¹ Jour. of Iron and Steel Inst., p. 318, 1908.

Assuming a correct polish these results are correct to the second decimal figure.

No. 1C was annealed for sixteen hours from 900° C. and is a segregated piece. No. 2C was quenched from 900° C., reheated to 600° C. and held at this temperature for twelve hours. No. 3C was quenched from 900° C. and reheated to 340° C. and held at this point for twelve hours. No 4C was quenched from 900° C., reheated to 200° C. and held at this temperature for eighteen hours. This reheating does not affect the structure to a noticeable degree, yet it softens the steel sufficiently to allow cutting. No. 5C was held at 1000° C. for thirty minutes and then quenched. No. 1D was held at 600° C. for twenty-four hours. No. 2D was quenched and reannealed, the exact time and temperatures not being known. Microscopic examination showed that it belonged more to class III. than to class II. No. 3D was cooled from 1000° C. to 200° C. during sixteen hours. This is a good specimen of perlite. No. 4D was quenched from 900° C. and heated to 340° C. No. 5D was held at 950° C. for thirty minutes and quenched in ice water.

Nos. 1C and 1D belong to class I. Nos. 2C and 3D to class II., Nos. 3C, 2D and 4D to class III., Nos. 4C, 5C and 5D to class IV.

Drude's¹ results are given for comparison.

DISCUSSION OF RESULTS.

It is seen that the results of Drude differ from those given here and that the difference is much larger than the error of observation. The difference lies mainly in the values obtained for principal azimuth, and this value is affected by scratches and not so much by the purity of the surface. Impurity of the surface, that is, a surface film, tends to decrease the value of principal incidence, and these results do not indicate any such impurity, the principal incidence in all cases being larger than that of Drude. From previous work² by the author on the optical constants of iron and nickel alloys the curve plotted between n and the per cent. of iron intersects the iron axis at a value of n equal to 2.48, and the curve between k and the per cent. of iron could be drawn to the iron axis at a point giving the value for k of 1.31. This leads to the conclusion that the iron observed by Drude contained some other metal, or else that there is some constant error in the method of polishing used here. But if for any reason these values of n are too high and of k too low they are consistently so, and the difference between the different steels exists independent of the absolute values of the optical constants.

¹ Ann. der Phys., 39, 481, 1890.

² Phys. Rev., 33, p. 453, 1911.

The fact that classes I. and II. give the same values for n and k proves that the light reflected from the large particles of the carbides of iron combines with that from the pure iron and gives light of the same form as is obtained from the small particles and the pure iron. Since this proves that the size of the particles up to this extent has no effect and since classes III. and IV. have the same values for n (and this value is different from that of classes I. and II.) the conclusion is that the carbides of iron have some effect on the iron molecule in both the hard steel and the troostitic form. This furnishes good evidence in support of the theory held by Campbell mentioned previously.

The decrease in k caused by an addition of carbon is so small that a change caused by heat treatment cannot be observed. But the changes in n caused by changes in carbon content and by heat treatments are quite large. Hence lack of similarity in these two factors could easily account for any lack of agreement between measurements of the optical constants of steels of unknown compositions and heat treatments.

In conclusion the author wishes to express his thanks to Prof. C. E. Mendenhall, of the University of Wisconsin, for assistance in the first part of this work, to Prof. E. D. Campbell, of the department of chemistry of this university, for suggestions and for specimens used in the work on steel, to Mr. A. E. White for assistance on the micro-photographs, and to Prof. K. E. Guthe for the apparatus used and for his interest in the progress of the work.

PHYSICAL LABORATORY, UNIVERSITY OF MICHIGAN,
June, 1912.

THE KERR ROTATION FOR TRANSVERSE MAGNETIC FIELDS,
AND THE EXPERIMENTAL VERIFICATION OF WIND'S
MAGNETO-OPTIC THEORY.¹

BY L. R. INGERSOLL.

Introduction.—There may be distinguished three different types of phenomena—collectively referred to as the “Kerr effect,” although Kerr himself was able to observe only the first two—which arise when a surface of iron, nickel, or cobalt, on which is incident plane polarized light, is magnetized. The three types correspond to the three possible cases of magnetization, viz., with the lines of force normal to the reflecting surface, or lying in the surface and respectively parallel or perpendicular to the plane of incidence of the light. The phenomena in general are somewhat complicated inasmuch as they are superposed on the optical effects which accompany the reflection of light from the unmagnetized surface. In the simplest and best known case—the reflection at normal incidence from a surface at which the lines of force are also perpendicular—the effect of magnetization is practically nothing but a simple rotation of the plane of polarization; in other cases the phase change and accompanying ellipticity, which is very weak in this simple case, become prominent. While a number of experimenters have made measurements for the first two types, the third case, *i. e.*, where the magnetic lines of force are normal to the plane of incidence—the so-called “equatorial” magnetization²—has, because of the minuteness of the effect, heretofore escaped observation with the exception of a single determination for iron made by Zeeman.³

The writer was led to take up the present work by a brief study of the effects of magnetization on the optical constants of the magnetic metals. The method which had been developed and used for determining optical constants, particularly in the infra-red spectrum,⁴ was found to be capable of such accuracy that it would seem to be possible as well as desirable to study the effect of various directions of magnetization on the azimuth of the reflected light. It was soon realized, however, that while observations of such azimuth changes were easily obtained they were in general

¹ This and the correlated work has been materially aided by grants from the Rumford Fund of the American Academy of Arts and Sciences.

² See Voigt, “Magneto- und Electrooptik, p. 277.

³ Arch. Neer. (II)., 1, p. 221 (1898).

⁴ Astrophys. Jour., 32, p. 265 (1910).

difficult to interpret, for the problem under these conditions became the complicated one of the Kerr effect for oblique incidence. For the case of transverse or "equatorial" magnetization, however, the results were found to be of peculiar interest because of the closeness with which they followed the predictions made by the late C. H. Wind¹ in his magneto-optic theory, and it is this experimental work which will be described.

Wind's Theory.—This explanation of the Kerr effect shares with those of other physicists of the Leyden school many of the characteristics of the Lorentz theory, in particular the idea of a new vibration component introduced by the magnetic field. Its interest from our standpoint lies in the completeness with which its author has worked out the case of magnetization perpendicular to the plane of incidence. He concludes that this should give rise to a minute difference of phase in the two component vibrations and to a small change in azimuth, this effect being due to the change in the component of the polarized light whose electric vector lies in the plane of incidence, *i. e.*, normal to the lines of force. The other component will not be altered. The effects which should be expected in the case of each of the three magnetic metals, for the wave-length of sodium light, have been worked out in detail, and the curves for the azimuth change—the phase change cannot be experimentally considered in this paper—are reproduced in Fig. 1. The azimuth change is not

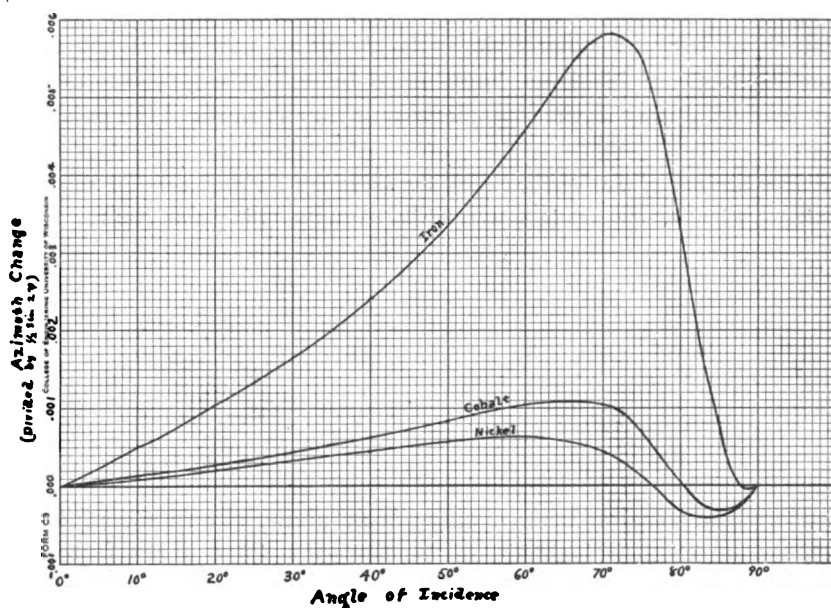


Fig. 1.

Wind's theoretical curves for azimuth change with transverse magnetization.

¹ Arch. Neer. (II.), I, p. 119 (1898).

directly expressed in these curves, but rather a quantity which is this change, measured in fractions of a radian, divided by $\frac{1}{2} \sin 2\psi$, where ψ is the "azimuth of restored plane polarization."

At the time of the appearance of Wind's theory no one had been able to detect an effect of this sort, but shortly afterward Zeeman succeeded in making observations in substantial agreement with the predictions of the theory both as regards azimuth and phase change, for the case which should yield a maximum effect, viz., iron at 75° incidence. So far as the writer is aware the subject has not been further handled experimentally up to the present time.

The present experiment may be outlined briefly as follows: Light from a Nernst glower (*G*, Fig. 2) is plane polarized and reflected from the surface (*S*) under trial. It will then in general be reduced to an elliptically polarized vibration, of which the two components are taken by passage through a large double image prism "analyzer" (*A*). The intensity of these two components can be compared for any wave-length by a special spectrophotometric apparatus.¹

While the ratio of these two components would enable one to find at once the "azimuth of restored plane polarization"—if this state of polarization is imagined restored by a suitable phase change—it is found much more convenient in practice to keep the two components of the same intensity by changing the azimuth of the incident polarized light. This azimuth of restored polarization—the *principal azimuth* if the corresponding angle of incidence happens to be the *principal incidence*—is then read off at once as the (complement of the) azimuth of the polarizing agent.

A comparison of this method with the ordinary Babinet scheme of determining optical constants is worth while to make this point clear. Fig. 3

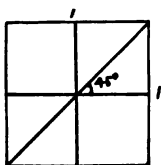


Fig. 3.

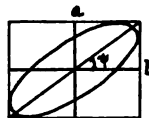


Fig. 4.

shows a plane polarized beam of azimuth 45° giving vibration components each unity, which is incident on the surface. Then if the factors by which the two amplitudes are changed are respectively *a* and *b*, Fig. 4 represents the vibration after reflection, the phase change being such

¹ For details of this arrangement see *Astrophys. Jour.*, 32, p. 265 (1910).

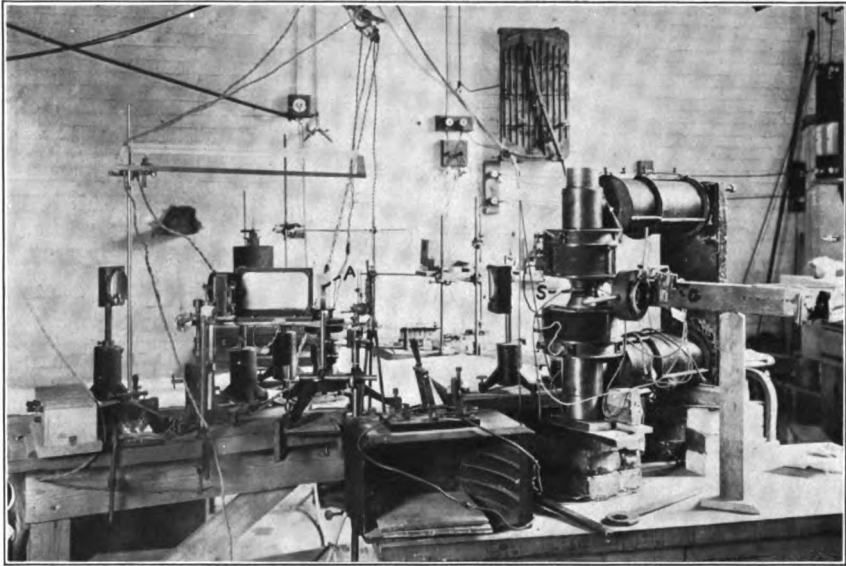


Fig. 2.
L. R. INGERSOLL

in general as to render it elliptical. If this elliptical vibration is then reduced by the Babinet compensator to plane polarized light the azimuth would be given by $\psi = \tan^{-1} b/a$. In the present method the component vibrations are equal after reflection (Fig. 6) and this evidently necessi-

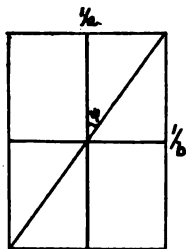


Fig. 5.

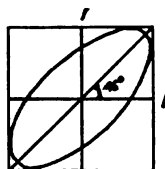


Fig. 6.

tates original components proportional to $1/a$ and $1/b$, so that the azimuth of the polarizer must be $90 - \psi$.

The magnetic field is produced by a powerful electromagnet, between whose poles the surface is held in a specially devised clamp. When the magnet is excited the azimuth ψ is changed to ψ^1 and the polarizer must be turned to azimuth $90 - \psi^1$ to make the two components equal again. The difference $\delta\psi$ is the change due to magnetization and is the quantity with which we are concerned. While it would also be of interest to measure the change of phase which accompanies this azimuth change, this would involve a complicated modification of the method which has not as yet been attempted, and which does not, at best, appear promising.

The procedure in making observations is slightly different from that indicated above but almost equally simple. As the change of azimuth on magnetization is a very small quantity—usually only a few minutes of arc—the galvanometer deflection corresponding to such change is determined as accurately as possible by repeated reversals of the magnet current. The deflection caused by a measured change of azimuth (*i. e.*, rotation) of the polarizer of, say, 15 minutes of arc is then noted, and the magnetic effect deduced by a simple proportionality. A correction is of course necessary for any direct action the magnet may have on the galvanometer.

Accuracy of the Measurements.—It will be readily understood that in work of this character—many of the angles of rotation are too small for even detection by the Babinet—a high degree of accuracy is not to be expected. Generally speaking the sources of error¹ are those incidental

¹ See papers by the writer in *Phil. Mag.* (6), 11, p. 51 (1906); *Phys. Rev.*, 23, p. 489 (1906); *Phil. Mag.* (6), 18, p. 86 (1909); *Astrophys. Jour.*, 32, p. 274 (1910).

to the writer's spectrobolometric method of studying polarization phenomena, but the chief source of error in the present case lay in the discrepancy which was frequently found between measurements made for the two azimuths lying symmetrically on each side of the normal. This was apparently due to a small component of the magnetic field perpendicular to the metal surface and hence could be eliminated by merely averaging the two results. This was done for the best observations, such as those for sodium light, and it is probable that the error in these measurements does not exceed a very few per cent.

RESULTS.

Two distinct series of observations were made with somewhat different experimental arrangements. In the first case parallel light was incident on the test surface while in the second the light converged to an image

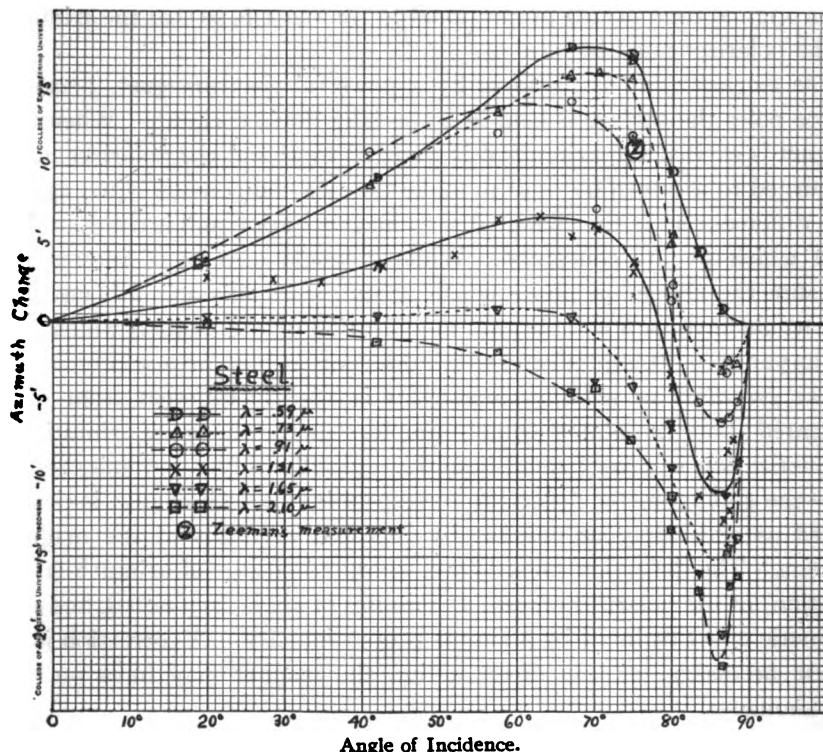


Fig. 7.

Azimuth curves for steel.

of the Nernst glower at this surface. The results were in substantial agreement. While the first method was theoretically preferable as giving a perfectly definite angle of incidence, the second had the decided ad-

vantage of affording much more light, especially for very oblique angles of incidence and as the extreme variation of incidence was hardly 3° the uncertainty from this source of error was not a serious objection and did not exceed a degree. The mirrors of steel, nickel, and cobalt were 5 cm. in diameter and about 5 mm. thick. They were optically plane and very perfectly polished, having been finished by Zeiss. Steel was used rather than iron because it takes a better polish. Its optical characteristics are very nearly the same as those of iron,—so much so that we shall not distinguish between them.

Measurements in the Visible Spectrum.—The observations were at first confined to about the same range of wave-length as that for which it had been found possible to measure the dispersion of these metals, viz., 0.8μ to 2.2μ ; but later it was found that when every precaution was taken to save light from losses at the various reflections, and to secure freedom from vibrations and other disturbances, it was possible to make very good observations even into the visible spectrum. These were carried out for the wave-length of sodium light (because of the low dispersion and wide slit necessary, the light really varied in wave-length from about 0.56μ to 0.62μ) and were particularly valuable as affording a direct check on Wind's predictions, as these were based on observations made only for this wave-length.

A general survey of the results may be obtained from Figs. 7 to 9, where practically all the determinations have been plotted as a function of the angle of incidence and for a number of wave-lengths. A positive azimuth change means that the azimuth is increased by the magnetic field under the following conditions: the lines of force running from left to right, as the observer sees the surface looking obliquely down on it in the direction in which the light is travelling. The values plotted are the doubled effects due to reversals of the magnet current.

While the consistency of the observations is not all that could be desired in many cases there is no chance for question as to the general form of the curves, viz., a zero value of the azimuth change for normal and for grazing incidence, and in general a positive maximum followed by a negative maximum for very oblique incidence. The general effect of an increase in wave-length is clearly to decrease the positive part and increase the negative part of the curve. By making measurements on the separate components it was found moreover that the only one affected was that vibrating in the plane of incidence or normal to the field, as predicted by theory.

For a more careful analysis of the results and a comparison with theory we shall consider only the measurements for the wave-length of sodium

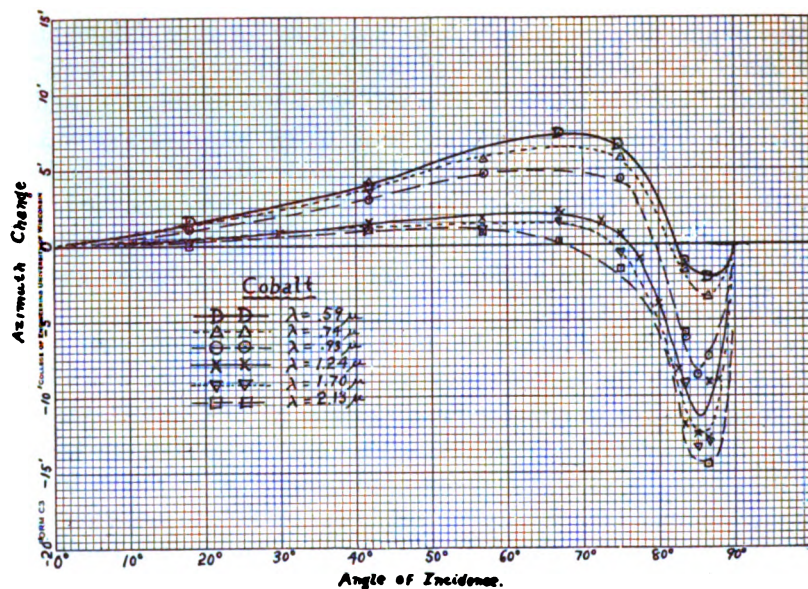


Fig. 8.
Curves for cobalt.

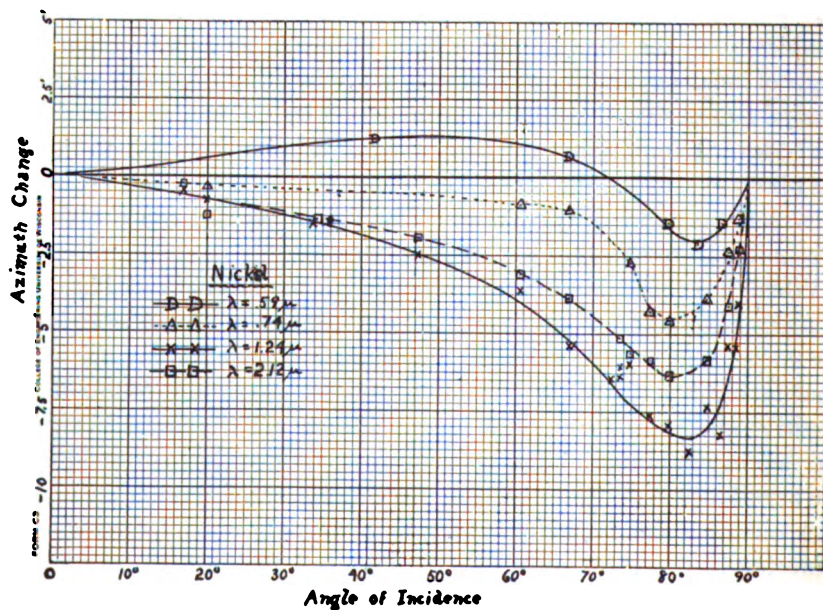


Fig. 9.
Curves for nickel.

light. A comparison of the curves of Fig. 1 with the corresponding ones in Figs. 7 to 9 shows at once the qualitative agreement of the results with Wind's theory, but for a more careful examination they must first be reduced to the same scale.

To do this Wind's results are first changed from circular measure to minutes of arc and then multiplied by the factor $\frac{1}{2} \sin 2\psi$. The angle ψ is readily computed for any angle of incidence from the equations given by

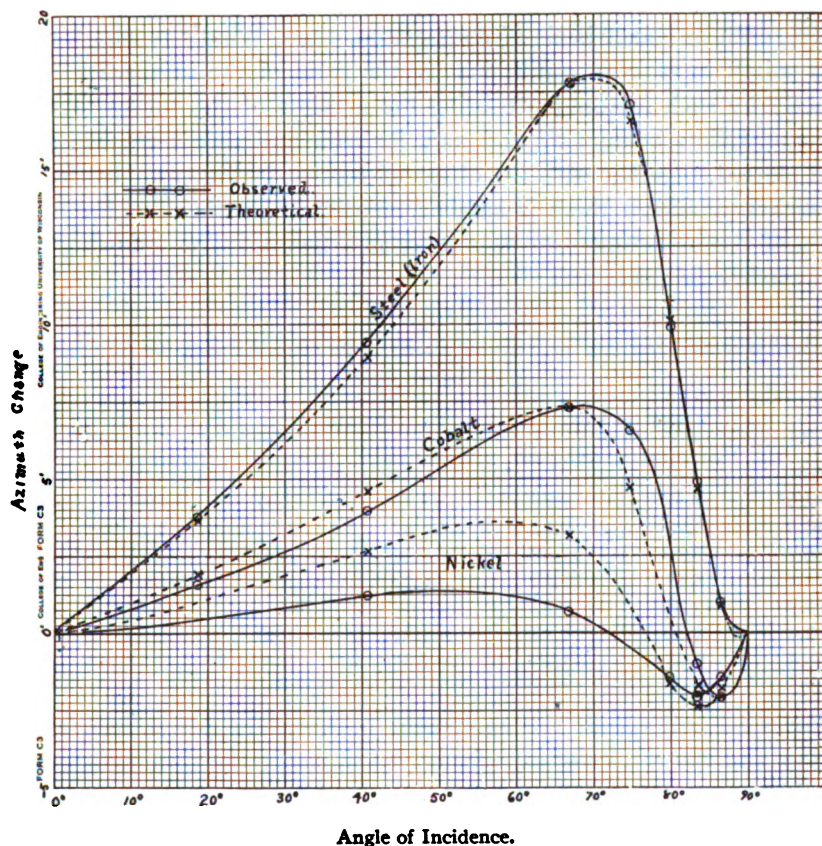


Fig. 10.

Theoretical and observed curves for the azimuth change accompanying transverse (equatorial) magnetization, for the wave-length of sodium light.

Drude (Optik, p. 344), Drude's values of the optical constants being used in this connection. The results (Wind's) are then doubled since our measurements are for reversal of the magnetic field.

The theoretical results would then be ready for comparison with the observations if they were computed for the same intensity of magnetization. This, however, is not quite the case. Wind's calculations for

iron are based on observations made for an intensity of magnetization of 1,400, while in the present case the induction was found by measurement to be 25,000 lines, which for this particular specimen—about .8 per cent. carbon—means an intensity of magnetization of perhaps 1,500. For cobalt the measured induction of 14,000 would mean an intensity two or three times that of Wind's which was only 430. For nickel the induction rather than the intensity was specified, and was given as 8,000 lines, while that in the present case was about 14,000.

To reach an exact basis of comparison, then, Wind's results for iron and cobalt have been multiplied by the ratio of the intensities—more exactly by such factors as to bring the maxima of the curves together, the multipliers being 1.07 and 2.08 for iron and cobalt respectively—and plotted, together with the observed results, in Fig. 10. In the case of nickel, although there is smaller theoretical justification for it, the ratio of the inductions has been used as a factor. The following table summarizes the results:

TABLE I.

Comparison of the Theoretical and Observed Values of the Azimuth Change for the Wave-length of Sodium Light.

Incidence.	Steel (or Iron).		Cobalt.		Nickel.	
	Theoretical.	Observed.	Theoretical.	Observed.	Theoretical.	Obs.
18.6°	3.67'	3.77'	1.93'	1.60'	—	—
40.8	8.98	9.42	4.62	3.98	2.68'	1.24'
66.9	17.80	17.80	7.30	7.30	3.18	0.70
74.6	16.57	17.11	4.73	6.56	—	—
79.9	10.08	9.87	—	—	-1.62	-1.41
83.4	4.70	4.88	-1.71	-1.00	-2.36	-2.09
86.4	0.88	1.00	-2.08	-2.07	-1.76	-1.40

For steel and cobalt the agreement between the predictions of this theory and the results of experiment must, under the circumstances, be considered as very good. In the case of nickel the agreement is only qualitative, making perhaps another instance in which this metal exhibits optical characteristics distinct from the other two magnetic metals.¹

Experiments with Heusler's Alloy.—Over half a dozen years ago the writer² found that the Kerr effect (normal incidence and normal magnetization) was apparently absent in the Heusler alloy, despite its other magnetic properties. As this result has been rather widely quoted and was, moreover, based on experiments with the earlier and cruder forms of the writer's apparatus, it seemed worth while to try Heusler's metal

¹ Phil. Mag. (6), 11, p. 72 (1906).

² Ibid.

in connection with the present work. The piece selected was a different specimen from that formerly used and was very strongly magnetic. The results are given in the accompanying table, and also the approximate azimuth changes for steel under the same conditions.

TABLE II.

Wave-length.	Incidence.	Azimuth Changes.	
		Heusler Alloy.	Steel.
.93 μ	40.8°	0.005'	10.2'
1.24	66.9	-0.015'	6.8'
1.70	84.4	0.003'	-15'

Similar measurements for normal (or polar) magnetization and an incidence of about 30° gave a possible rotation for this same specimen of alloy not to exceed one or two tenths of a minute. We may then safely conclude that magneto-optic effects of this sort, if they exist at all in Heusler's metal, are less than a hundredth as large as the corresponding effects in steel.

PHYSICAL LABORATORY,
UNIVERSITY OF WISCONSIN,
July, 1912.

NEW BOOKS.

- The Energy System of Matter.* By JAMES WEIR. New York: Longmans, Green and Co., 1912. Pp. vi + 200.
- Per-Acids and their Salts.* By T. SLATER PRICE. New York: Longmans, Green and Co., 1912. Pp. vi + 123.
- The Mechanics of the Aeroplane.* By CAPTAIN DUCHÊNE. Translated from the French by JOHN H. LEDEBOER and T. O'B. HUBBARD. New York: Longmans, Green and Co., 1912. Pp. x + 231. Price, \$2.25.
- Electrical Injuries.* By CHARLES A. LAUFFER. New York: John Wiley and Sons, 1912. Pp. vi + 77. Price, \$.50.
- General Physics.* By W. WATSON. New York: Longmans, Green and Co., 1912. Pp. xiii + 564.
- The Early Naturalists, Their Lives and Work.* By L. C. MIALL. New York: Macmillan Co., 1912. Pp. xi + 396. Price, \$3.50.
- Précis d'Optique.* By PAUL DRUDE. Paris: Gauthier-Villars, 1912. Pp. 1 + 362. Tome II. Price, 12 fr.
- Notes sur la Physique et la Thermodynamique.* By E. H. AMAGAT. Paris: A. Hermann and Fils, 1912. Pp. ii + 146. Price, 6 fr.
- Elements of Hydrostatics.* By GEORGE W. PARKER. New York: Longmans, Green and Co., 1912. Pp. viii + 150.
- Réception des Signaux Radiotélégraphiques Transmis par la Tour Eiffel.* Paris: Gauthier-Villars, 1912. Pp. 1 + 56. Price, 1 fr.
- Das Relativitätsprinzip.* By A. BRILL. Leipzig: Teubner, 1912. Pp. 1 + 29. Price, M. 1.20.
- Mémoires sur l'Électricité et l'Optique.* By A. POTIER. Paris: Gauthier-Villars, 1912. Pp. xx + 330. Price, 13 fr.
- Contributions from the Jefferson Physical Laboratory of Harvard University for the Year 1911.* Vol. IX. Cambridge, Mass.
- Der Aether.* By M. LA ROSA. Translated from the Italian manuscript by K. MUTH. Leipzig: Barth, 1912. Pp. 1 + 116. Price, M. 2.50.
- Tables Annuelles de Constantes et Données Numériques de Chimie, de Physique et de Technologie.* Chicago: University of Chicago Press, 1912. Price, \$6.00.
- Electricity and Magnetism.* By S. G. STARLING. New York: Longmans, Green and Co., 1912. Pp. vi + 583.
- A Text-Book of Physics.* By A. WILMER DUFF. Third Edition Revised. Philadelphia: Blakiston's Son and Co., 1912. Pp. xvi + 686.
- Physik. Zum Gebrauch bei physikalischen Vorlesungen in höheren Lehranstalten sowie zum Selbstunterricht.* By PROF. DR. H. BÖTTGER. Braunschweig: Vieweg und Sohn, 1912. Pp. xiii + 983. Price, M. 15.

THE PHYSICAL REVIEW.

ON ELECTROMAGNETIC INDUCTION AND RELATIVE MOTION.

BY S. J. BARNETT.

THIS investigation originated in 1902 with an attempt to develop a method for the solution of the old problem of unipolar induction. In its best known form this problem may be stated as follows: A magnet symmetrical about its axis and rotating about this axis uniformly is touched in two points *A* and *B* not in the same equatorial plane by the ends *A* and *B* of a wire *ACB*. A steady current traverses the circuit thus formed. Which is the seat of the electromotive force, the wire or the magnet? That is, do the lines of magnetic induction rotate with the magnet and cut the external conductor *ACB*; or do the lines, like the external conductor, remain fixed while the magnet rotates through them? Faraday thought that his experiments proved the second alternative, and Pluecker shared his views, while Weber disagreed. It was first shown by S. Tolver Preston¹ that the experimental results could, as is now well known, be explained equally well on either hypothesis.

Preston went further and adduced cogent arguments in support of the first alternative, which, in a later paper,² he quotes Lord Rayleigh as also favoring. In the first place, Faraday had admitted that a magnet in translation carried its lines of induction with it; and Preston called attention to the fact that translation is involved in all rotation (rotation, according to his statement, being a particular case of translation); thus a large magnet in rotation is made up of many small magnets in translation, though the translation is in curved lines. In the second place if, instead of rotating the magnet, we keep the magnet fixed and rotate the conductor *ACB* in the opposite direction with the same speed, we obtain the same electromotive force *and its seat is in the conductor ACB*. Now

¹ S. Tolver Preston, *Phil. Mag.* (5), 19, 1885, p. 131.

² S. Tolver Preston, *Phil. Mag.* (5), 19, 1885, p. 215.

the *relative motion* between the parts of the system of magnet and wire is exactly the same in the two experiments, hence we should expect *ACB* to be the seat of the electromotive force when the magnet rotates and the wire remains fixed. A third argument, relating to the similarity between an electric coil and a magnet, cannot be accepted as valid, inasmuch as we know no more about the phenomena of unipolar induction in the case of an electric coil rotating without iron than we do in the case of a magnet. Both sets of phenomena can be explained equally well in the same way on either hypothesis.

Lecher¹ was not satisfied that Preston's argument was conclusive and published some highly interesting experiments, also made with closed circuits, in 1895. Although he regarded these experiments as favoring Faraday's view, they cannot be considered as establishing it. Lecher also attempted to solve the problem by electrostatic methods, but without success.

In 1902 it occurred to me that the problem could be solved by the following method. A cylindrical condenser is placed in an approximately uniform magnetic field parallel to its axis, and the magnets producing the field are rotated while the condenser is short-circuited by a wire at rest like itself. While the magnets are still rotating, the connection between the armatures of the condenser is broken; and the condenser is tested for charge after the field is annulled, or the rotation stopped, or the condenser removed. It was argued that if the lines of induction moved with the magnets the condenser would receive charges which could be computed, and that it would remain uncharged in case the lines remained fixed and the magnets moved through them. It was found later that a somewhat similar idea, discussed below, had previously occurred to Preston;² and in 1908 Mr. Tracy D. Waring³ proposed an experiment essentially the same in principle as that which had occurred to me. As shown below, however, our reasoning was erroneous. Since what precedes was written, there has appeared an experimental paper on the subject by Mr. E. H. Kennard,⁴ which also is based on incorrect theory.

¹ E. Lecher, Wied. Ann., 54, 1895, p. 276.

² S. Tolver Preston, Phil. Mag. (5), 31, 1891, p. 100.

³ Tracy D. Waring, Trans. Am. Inst. Elec. Eng., 27, 1908, p. 1366.

⁴ E. H. Kennard, Phil. Mag. (6), 23, 1912, p. 937. The inconclusive character of Mr. Kennard's work will be apparent after reading the theoretical part of this article. Using a method resembling my own, but with the iron core of the electromagnet rotating alone while the magnetizing coil remained fixed like the condenser, he thought he had proved that the lines of induction did not move with the iron, because the electrometer received no charge in his experiments. This conclusion, as shown here, does not follow. It should be pointed out also that Mr. Kennard's calculation of the effect to be expected on the moving line hypothesis assumes that on this hypothesis all the lines in his experiment would move with the iron;

Preston imagined a magnet NS , Fig. 1, rotating about its axis near a flat circular metal plate K ; and concluded that if the lines of induction remained fixed the magnet would become charged by the motional electromotive forces within it, and the plate K by electrostatic induction, as indicated in (a). And he concluded that on the other hypothesis the plate would be charged as in (b), the magnet being uncharged except to a slight extent by induction. Preston consulted Hertz with reference to the matter; and Hertz "laid stress on the great interest attaching to

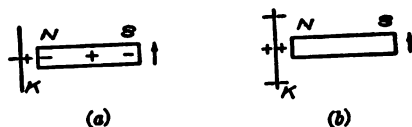


Fig. 1.

the inquiry, and agreed that the conditions of test as proposed should be capable of deciding this question"; but he informed Preston that "with a normally sensitive electrometer a considerable velocity of rotation with a magnet of large size would be calculably required to produce a distinct deflection under either hypothesis." And Preston stated that "facts of this kind discouraged from trying the experiment." At the same time he expressed hope for the production in the future of a much more sensitive electrometer, with a quartz suspension.

Equally, however, with those who have tried to solve the problem by experiments on closed circuits, all who have proposed electrostatic tests to decide between the two hypotheses have, if we admit the validity of current electromagnetic theory, fallen into error; for it is not difficult to show that the total electric intensity and the electric density are in all cases the same on the two hypotheses. The error has arisen through failure to take proper account of the motional intensity in the dielectric.

For the special cylindrical condenser system considered here and imagined to be placed in a uniform magnetic field, whose direction is parallel to its axis and which is produced by a coil or magnet system in rotation, the electric field may be investigated as follows. The length of each cylinder will be considered great in comparison with the distance between

whereas it is impossible to say what fraction, on this hypothesis, would adhere to the iron and move, and what fraction would adhere to the coil and remain at rest. On account of the fact that a part of his field-producing system was at rest, instead of all being in motion, the experiments are also inconclusive as to the important matter of relative motion discussed below. To make them of value it would be necessary to make experiments on the charge taken by a short-circuited condenser, or the motional electromotive force developed in a Faraday disc, rigidly attached to the magnetizing coil of an electro-magnet whose core remains fixed while the coil rotates with the disc. The condenser experiments would be difficult; successful experiments with the disc would, apparently, be impossible.

the two cylinders, so that end effects will be negligible, and the outer cylinder may be supposed longer than the inner and closed by conductors at the ends. A fine wire may be supposed to connect the two armatures.

On the hypothesis of fixed lines it is evident that there will be no intensity, either field or motional, anywhere within the condenser system.

To obtain the field on the hypothesis of moving lines, imagine first that no motional intensity is produced in the dielectric. Then if ψ denotes the electromotive force produced in the wire by the motion of the lines of induction across it, the condenser will be charged to the voltage $V = -\psi$; and if r' and r'' denote the radii of the opposing faces of the internal and external armatures, the field intensity E , now the total intensity, at any point within the condenser and distant r from the axis will be

$$E = \frac{V}{r \log r''/r'} = - \frac{\psi}{r \log r''/r'}.$$

Now imagine that no motional intensity is produced in the wire, and consider the intensity produced by the motion of the lines of induction through the dielectric. The total intensity will be the same as if the field and wire remained fixed and the *whole* dielectric rotated in the opposite direction, and will be given by the equation¹

$$f = e + E = \frac{\omega B}{r} \left(\frac{r''^2 - r'^2}{2 \log r''/r'} \right) = \frac{\psi}{r \log r''/r'}.$$

In the actual case the two effects will be superposed, so that the total intensity will be zero at every point. The same thing is true of the total intensity within the inner armature.

On both hypotheses, therefore, the condenser system remains uncharged.² The result is entirely independent of the magnitude of the dielectric constant.³

To establish the general proposition,⁴ consider a magnet NS and any system of conductors such as a metal disc K , arranged according to Preston's idea, Fig. 2. First imagine the magnet to start into rotation

¹ S. J. Barnett, *Phys. Rev.*, 27, 1908, p. 432. The above result follows at once by adding the motional intensity ωBr to the field intensity given by equation (11), $(K-1)/K$ being appropriately replaced by unity.

² That the charge is zero also follows at once from the consideration that the dielectric and short-circuiting wire together form a closed circuit, so that the electromotive force induced in the wire is just equal and opposite to that induced in the dielectric.

³ See S. J. Barnett, *Ann. der Phys.*, 30, 1909, p. 416.

⁴ This subject, as I found after independently developing the theory for my own experiments several years ago, had been previously treated by Poincaré (*L'Eclairage Electrique*, 23, 1900, p. 41) and by M. Abraham (*Theorie der Elektrizität*, Vol. I., 1907, p. 418); but neither treatment seems to me satisfactory.

while the lines of induction remain fixed. Then at any point P within the magnet there is a motional electric intensity $e = [vB]$, where v denotes the velocity and B the induction at P . Owing to this intensity a transient current exists in the magnet and there is developed in and about it an electrical field which becomes steady as the motion becomes uniform. In the steady state the field intensity E at P is equal and opposite to the motional intensity e , the total intensity $f = e + E$ being zero. In the conductor K (at rest)¹ each intensity is zero. In the dielectric the field intensity E exists alone. The potential difference V , equal to the line integral of the intensity E , from any point B to any point A on the surface of the magnet along any line whatever connecting the two points is equal to the line integral ψ of e along any line ADB from A to B within the magnet. The electric density, which is proportional to the divergence of the total intensity f , is zero everywhere except at the interfaces between the conductors and the dielectric.

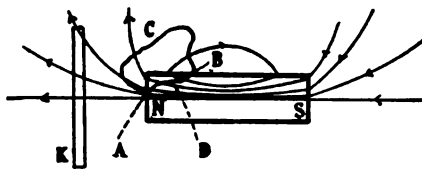


Fig. 2.

If, on the other hand, the lines of induction move as if rigidly attached to the magnet, e , E , and $f = e + E$ are all zero within the magnet; in the conductor K the total intensity f is zero, so that $E = -e$; while in the dielectric the total intensity is $f = e + E$. And this total intensity at any point is precisely the same as on the other hypothesis; for the integral of f along the line BCA is equal to the integral of e along the same line, since the integral of E vanishes along BDA and therefore along BCA , and this integral of e is equal to ψ . As A and B are any points on the magnet and ACB any line of any shape whatever connecting them, it follows that the total intensity f at any point is the same on the two hypotheses. Hence, if the *whole* dielectric is cut by the moving lines, so that the motional intensity, as well as the field intensity, acts on the whole dielectric, the electric density everywhere is exactly the same as before.

If the lines of induction neither remain at rest nor move with the full angular velocity of the magnet, the result, so far as f and the charges are concerned, is precisely the same. Though e and E will differ according to the hypothesis made, we shall always have $E = -e$, or $E + e = 0$, in the conductors, and $f = e + E$ in the dielectric, $E + e$ at any point being independent of the hypothesis.

¹ The proposition is easily demonstrated, by the application of the same method, for the case in which one or more conductors of the system K , the material dielectric, or the æther, are in motion.

Moreover, if we reject the hypothesis of the æther, as existing independently of the electromagnetic field, the result is the same. For to the electric displacement produced by the motional intensity acting on the material dielectric must be added that which would, on current theory, accompany the lines of magnetic induction if they moved in space otherwise free or filled with radiation.

Hence all attempts to solve the problem of unipolar induction by the measurement of charges or total intensities or electromotive forces are vain. For the same reason the problem cannot be solved by measuring the force upon a charged body in the field of a rotating magnet, as in an experiment proposed by Sir Oliver Lodge,¹ and suggested by Mr. Waring² as a means of solving this problem. It appears that the only way in which the problem can be solved electrically is to measure the two intensities separately, and how this can be done is not evident.

Much importance still attaches, however, to the proper execution of the experiment described above. For there can be no doubt, from the experiments of Faraday and many others, that if the short-circuited condenser is rotated about its axis in the fixed magnetic field it becomes charged by the motional electromotive force in the short-circuiting wire; while if the condenser and its short-circuiting wire remain fixed and the magnet or electric coil producing the field is rotated with the same speed in the opposite direction, the relative motion between the two systems is exactly the same as before, and one would expect the same charges to be developed. Indeed, the fact that the electromotive force induced in a closed circuit by relative motion between it and a magnet is independent of the one that moves is one of the considerations that led Einstein to the principle of relativity. Furthermore, a positive result would both reveal a flaw in current electro-magnetic theory and confirm Preston's idea on the seat of the electromotive force in unipolar induction. I have therefore carried to its completion the investigation begun with a different idea long ago.

After a few preliminary experiments in 1904, in which the whole condenser was short-circuited in the magnetic field and moved out after insulation to the electrometer connections, improved apparatus, in which only the outer armature moved, was tried several years later, but with inconclusive results. Work on the problem was resumed this year, and with the apparatus described below it has been possible to obtain precise and conclusive results.

Although the phenomena of unipolar induction hitherto known are

¹ Oliver Lodge, *Modern Views of Electricity*, 1907, Sec. 73. See also *Phil. Mag.* (5), 27, 1889, p. 469.

² Tracy D. Waring, *loc. cit.*

essentially the same whether a rotating magnet or a rotating electric coil produces the magnetic field,¹ it was thought worth while to investigate the behavior of both. The first experiments in the recent work were on fields without iron.

A simplified diagram of apparatus is given in Fig. 3. *C*, the coil which produced the magnetic field, was wound from about 5,000 turns of No. 14

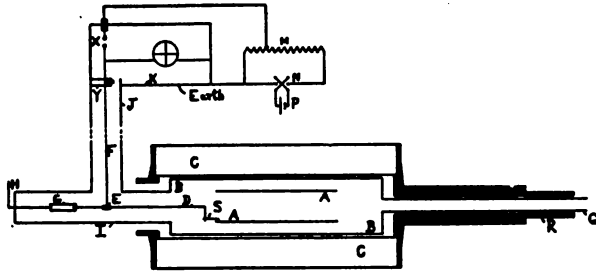


Fig. 3.

DCC copper wire on a substantial brass bobbin, to whose ends were fitted hollow axles of brass and bronze. The outer armature *B* of the (air) condenser was a brass tube 28 cm. long and 6.67 cm. in internal diameter. It was continuous with the brass tubes *Q*, *I*, and *J* and the brass lining of the wooden electrometer and key boxes *K*. The inner armature *A* was a brass tube 14.9 cm. long and 3.97 cm. in external diameter, mounted coaxially with the coil and with *B* at their common center, and insulated from *B* with two small amber blocks. To one end of *A* was attached a straight wire *S*. This wire, together with the bent wire *D*, which passed through a brass sleeve *E* in an amber support and was operated by the bent wire *H*, insulated from it by the amber block *G*, formed a key for connecting *A* by the wire *F* to the electrometer and its key *X*. The construction of this key *X* is illustrated in Fig. 4, except for a small lever

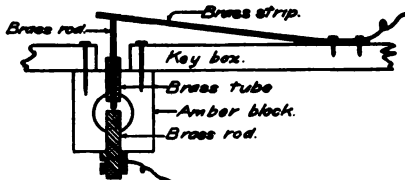


Fig. 4.

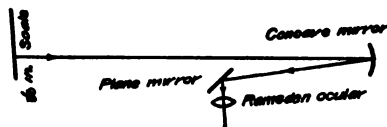


Fig. 5.

by the operation of which the key could be closed. Voltages for standardizing the apparatus were supplied by a dry cell *P* and a 10,000 ohm universal shunt box *M*, to which the cell was connected through the reversing switch *N*. The moving system was mounted in heavy brass bearings

¹ See especially O. Grottrian, *Ann. der Physik*, 10, 1903, p. 270.

screwed to a heavy wooden board, which was bolted to the cement floor. It was driven by an electric motor at measured speeds in the neighborhood of 20 revolutions per second and supplied with measured currents in the neighborhood of 20 amperes. The method of procedure made it necessary for the current to traverse the coil for only a few seconds in the several minutes required for each complete observation. The fixed system *QBI* was supported on the same base which carried the moving system. The electrometer was a modified Dolezalek instrument mounted on a wall which was absolutely free from appreciable vibration. The optical system made up a sort of Newtonian telescope, as shown in Fig. 5, and made it possible to read tenths of divisions on a one fiftieth inch scale $2\frac{1}{4}$ meters from the mirror. Sensibilities up to more than 10,000 divisions (double deflection) per volt were used. A slow stream of dried and filtered air was passed almost continuously through the tube *Q*, the condenser system *AB*, the tubes *I* and *J*, and the electrometer and key boxes *K*. To keep the condenser system cool during observations a rapid stream of air was ordinarily passed through the space between *QBI* and the coil *C*.

If the space between the tubes *A* and *B* were traversed by a uniform magnetic flux ϕ , and the short-circuited condenser system instead of the coil were rotated at a speed of n revolutions per second, the condenser would become charged to an electromotive force $n\phi$. The charge taken by the wire *DEF* would be very small in comparison with that taken by *AA* both in this case and in the actual case in which ϕ is not uniform. This charge does not enter into the calculations for experiments made with the key system described above, and is negligible in the later experiments made with the arrangement of keys described below.

To determine experimentally the electromotive force *E* to which the condenser is charged when it and the key system remain fixed while the coil rotates at the speed n revolutions per second, the procedure is as follows: The electrometer key *X* being closed and *N* open, the key *DS* is closed, thus connecting *A* and *B* together while the coil is in motion and the flux ϕ traverses the region between *A* and *B*. Then, in succession, contact between *D* and *S* is broken, the field is annulled and the motor switch opened while *X* is opened¹ and finally *DS* again closed.

¹ If the electrometer needle is deflected by the magnetic field of the coil or magnets, it is important that *X* be left closed until the needle has come to rest after the field is annulled. Otherwise a portion of the charge bound by the needle in its deflected position will be released to the insulated system as the needle approaches its zero, and there will remain a deflection which changes sign with the current. This effect, being independent of the rotation, is easily eliminated. This was done in some of the earlier work with iron when sufficient precision had been attained to detect the effect; but in all the final work, both with and without iron, the needle was allowed to come to rest before opening the key.

The reading of the electrometer is then taken. The process is repeated with the current in the coil reversed. If d denotes the double deflection of the electrometer, V the corresponding voltage, C the capacity of the system AB , and K the capacity of this system together with that of the electrometer and connections, we have

$$E = \frac{K}{C} \cdot V, \text{ or } V = \frac{C}{K} \cdot E = Ad,$$

where A is a constant. If d' denotes the double deflection corresponding to a known voltage V' , we have also

$$V' = Ad'.$$

Hence, if E were equal to $n\phi$, the double deflection D to be expected would be

$$D = n\phi \cdot \frac{C}{K} \cdot \frac{d'}{V'};$$

and the immediate aim of the experiment is to determine the ratio d/D .

If the axial flux between the cylinders is not uniform, the mean value $\bar{\phi}$ throughout the length of the inner armature may be substituted for ϕ in the above formula without introducing much error, provided that ϕ does not depart greatly from uniformity and that the capacity of the system AB per unit length is nearly constant. From the dimensions and arrangement of the armatures it is evident that the second condition was satisfied in these experiments, and measurements on the magnetic field showed that the *minimum* and *maximum* values of ϕ did not differ by more than about 5 per cent. The axial flux was investigated with five test coils (set coaxially within the electric coil) ranging in diameter from a little less than the outer diameter of A to a little greater than the inner diameter of B , a properly tested Hibbert magnetic standard, and a ballistic galvanometer. Observations were made with each coil for every centimeter of the length of the condenser. The mean axial intensity thus found for the whole region occupied by the condenser was 119 gaussers for a current of 1 ampere in the electric coil, the average departure from the mean given by the separate coils for the mean intensity within the cylinders described by them being about 1/6 per cent. This made $\bar{\phi}$ equal to 268×10 maxwells for a current of 1 ampere in the coil.

In order to eliminate errors arising from changes in the electrometer's sensibility and zero and from extraneous electromotive forces, the observations were made on a regular time schedule; the observations for d were repeated in inverse order; and the double deflections were obtained from the corresponding mean scale readings. If L denotes the mean

scale reading when the current traversed the coil in one direction, and R the reading when the current was opposite, $d = L - R$. The four observations necessary to obtain one value of $L - R$ ordinarily constituted one *set*.

Nine sets of rotation experiments made in the early part of the work indicated that the average value of $(L - R)/D$, with due respect to sign, was zero; but the results were very irregular, the average numerical value of $(L - R)/D$ being about one fifth.

Much of the trouble was traced to the operation of the keys X and DS , which were therefore replaced by the keys D and L , Fig. 6, of the type

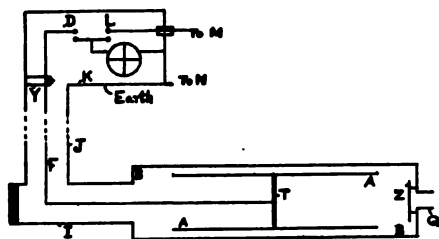


Fig. 6.

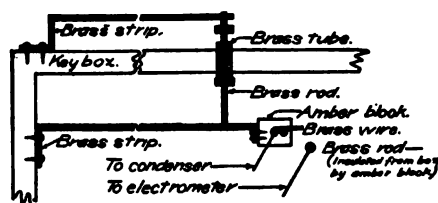


Fig. 7.

illustrated in Fig. 7, both placed in the key box. The difficulty with the old keys probably arose from the abrasion of fine particles of metal when the keys were operated, and from the presence of residual dust. Some of the discrepancies arose from the absence of perfect dryness, or constant conditions as to dryness, and some probably arose from the tremor given to the condenser and key DS by the rotation of the coil. The condenser, slightly altered, was therefore mounted directly on the wall; the amber insulators were all scraped, cleaned, and discharged in the hot gases from the flame of a bunsen burner; and the apparatus was more thoroughly washed out with filtered air. At the same time the speed was increased about 50 per cent. After these changes the discrepancies were greatly reduced, the average numerical value of $(L - R)/D$ in six sets being about 3 hundredths.

In the hope of securing still better results several changes were made. The keys were slightly improved and every effort was made to operate them with extreme regularity, the outer air stream was abolished, attempts were made to free the condenser and key box more completely from dust and to keep the air in a more nearly uniform state of dryness, and the sensitiveness of the apparatus was increased by reducing the capacity of the electrometer and connections and increasing the capacity of the condenser, and by increasing the scale distance to 3.4 meters. The scale could still be read to tenths of divisions.

The length of the outer tube BB of the new condenser, Fig. 6, was 28 cm. and its internal diameter 6.64 cm. The length of the inner tube AA was 20.0 cm. and its external diameter 5.08 cm. The capacity C_1 of this condenser was thus approximately 37.3 es. units.

The connecting wire F , with diameter 0.023 cm., had an effective length of 4 cm. (from the end of AA to the end of BB) in the tube BB and a length of 114 cm. in the tube IJ , with internal diameter 3.8 cm. The capacity of this system was thus approximately 11.5 es. units. The capacity of the key D (to a point just below the amber piece Y) was found by experiment to be 2.0 es. units. The capacity C_2 of the key and connecting system was thus approximately 13.5 es. units.

The capacity C_3 of the electrometer and permanent connections was determined by comparison with $C_1 + C_2 = 50.8$ es. units.

From the magnetic observations already described and additional observations made necessary by the increased length of the new condenser the mean value of ϕ , the flux through the cylindrical space between its two armatures, was found to be 169×10 maxwells for a current of 1 ampere in the coil. The maximum and minimum values of ϕ differed by about 10 per cent. of the maximum.

The coil was driven at a speed of about 32 revolutions per second, and was traversed by a current of 20 amperes.

Ten sets of observations were obtained. During the eight hours (including four hours intermission) within which the rotations occurred C_3 decreased from 60.4 to 56.5 es. units, the voltage sensibility varied from 12,900 to 12,800 divisions per volt, and D increased from 47 to 48 divisions. The average value of $(L - R)/D$ without respect to sign for the ten sets was 1.4 hundredths, and the average value with due respect to sign was -0.01 hundredth.

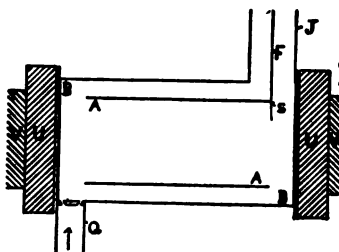


Fig. 8.

To investigate the matter when the magnetic field was produced by electromagnets the apparatus was suitably modified. Two large electromagnets UV , were substituted for the electric coil (Fig. 8). The cores were of steel shafting 7.5 cm. in diameter, and were capped with flat pole pieces UU , 12.3 cm. in diameter and 2.8 and 2.6 cm. thick, of wrought iron. The two electric coils surrounding the cores were similar, one of them being the coil C used in the experiments without iron. A was a brass cylinder 6.99 cm. in external diameter and 15.00 cm. long, B a brass cylinder 9.99 cm. in internal diameter and nearly 20 cm. long. The faces

of the poles UU were 20.6 cm. apart, and the whole system of cylinders, rotating and fixed, was mounted symmetrically about the center of the magnetic field. The cylinder B was covered with brass caps 12.3 cm. in diameter, and provided with a tube Q by which a stream of filtered air could be sent through the condenser and electrometer systems. The electromagnets were driven by an electric motor at about 32 revolutions per second. In the earlier part of the work the condenser system was supported on the same base boards with the electromagnets, and the key system consisted of the same key X used in the experiments without iron together with another key constructed on the same principle as DS in those experiments and similarly located; but experience of the sort already referred to in the case of the experiments without iron led to changes quite similar to the changes made in those experiments. The calibrating apparatus and the method of observation were the same as in the experiments without iron.

The mean value of the axial magnetic flux in the space between A and B was determined by ballistic observations as in the former experiments, but with two sets of test coils in the shape of two uniformly wound solenoids with the same number of turns per cm., one with diameter slightly less than the external diameter of A , the other with diameter slightly greater than the internal diameter of B , and each somewhat longer than A . Each solenoid was provided with four pairs of leads—two leads with 20 turns between them, two with 40, two with 60, and two with 68, making four test coils for each solenoid, all with a common center. Each solenoid in turn was placed symmetrically in the magnetic field and the flux through each of its four coils determined when a current of 10.0 amperes traversed the electric coils of the magnets. By plotting the difference between the fluxes for the corresponding coils as a function of the number of turns, obtaining from the curve the flux corresponding to the number 64.9, the number of turns in 15 cm., the length of A , and multiplying by the ratio of the right cross-section of the space between A and B to the right cross-section of the space between the mean circumferences of the solenoids, the value of $\bar{\phi}$ for a current of 10 amperes in the electric coils was found to be about 289×100 maxwells. Table I.

TABLE I.

Number of Turns in Each Coil.	20	40	60	68
Magnetic flux in maxwells $\times 10,000$	55	122	189	220

gives the relation between differential fluxes and turns for the solenoids

and shows that, as in the previous experiments, the flux ϕ is not far from uniform. Magnetic tests for currents above 10 amperes were carried out, but the comparatively small increase in flux (20 per cent. in changing from 9 to 16 amperes) did not appear to justify the large additional amount of heat generated in the coils; so that the experiments on rotation were always carried out with currents in the neighborhood of 10 amperes. Except in the early experiments, the currents were always somewhat *greater* than 10 amperes.

The earliest experiments were still less precise than the corresponding experiments without iron, but gave, like them a null effect. With the condenser on the wall and the key system improved, but not in its final state, the average magnitude of $(L - R)/D$ in 14 sets was reduced to about 5 hundredths. A slight further reduction to about 4 hundredths in 23 sets was made after the readjustment of the electrometer.

Finally, after still further attempts to improve conditions, 12 sets were obtained in which the average magnitude of $(L - R)/D$ was reduced to 1.7 hundredths, the average value with due respect to sign being $+ 1.2$ hundredths. During nearly all of the time in which the last five sets were obtained the electrometer was in unusually stable condition; before these sets were begun the clamp of one of the keys had been improved; and throughout the five sets the keys were operated with even greater care than formerly. For these sets the average magnitude of $(L - R)/D$ was 0.8 hundredth, and its value with due respect to sign was $+ 0.6$ hundredth. About an hour after the last set was obtained rotations were again started; but conditions had become poor, giving large discrepancies, and observations were discontinued.

In the interval of four hours devoted to these 12 sets the voltage sensibility of the electrometer ranged from 12,700 to 12,600 divisions per volt; the capacity C_2 of the electrometer decreased from 54.4 to 50.3 es. units; and D changed from 28 divisions to 29 divisions.

The capacity of the condenser AB was approximately 21.0 es. units; the capacity of the wire (about 106 cm. in length, 0.023 cm. in diameter) and tube $SFY - J$ was about 10.4 es. units; and that of the key DY , about 2.0 es. units, (Figs. 6 and 8). C_3 , the capacity of the key and connecting system, was thus approximately 12.4 es. units. C_3 was again obtained by comparison with $C_1 + C_2$.

The investigation leads conclusively to the result that the condenser system, when it remains at rest and the system producing the field rotates, receives not more than a minute fraction of the charge it would receive for the same relative motion if the system producing the field remained at rest. Within the limits of error of the experiments—about

1.4 per cent. in the experiments without iron, and about 1 per cent. in the experiments with iron—the fraction is zero.

For the construction of most of the special apparatus used in this investigation I am indebted to Mr. Tudor T. Hall, mechanician at the Tulane University of Louisiana, and Mr. Arthur Freund, mechanician in this laboratory. I am indebted to Mrs. Barnett for important assistance throughout the experimental part of the work.

THE PHYSICAL LABORATORY,
THE OHIO STATE UNIVERSITY,
July and October, 1912.

ON ELECTROSTATIC SCREENING BY THIN SILVER FILMS.

BY MISS SHIRLEY HYATT.

THE original purpose of this investigation was to determine the thickness of a metallic film necessary to screen an electroscope within a closed, hollow conductor from induction by outside charges. It had been observed that a glass vessel covered with a thin silver film was transparent to induction from outside charges, and the present work was undertaken to determine the limiting thickness through which the effect could be observed.

The first part of this work was done in the year 1907-08. All the films used in the investigation were chemically deposited upon glass plates by the Rochelle salts method, as described in Mann's Optics. A tin can with small windows cut in opposite sides was first used as a hollow conductor. The windows were screened by wire gauze, which was soldered to the can. A gold leaf electroscope was placed in the can with its charging disc about two millimeters below the cover of the can. A hole a little larger in circumference than the disc of the electroscope was cut in the cover of the can. Over this hole the silvered plate was placed, the film side resting on the tin, thus making a closed, hollow conductor of the can. The leaves of the electroscope were observed by means of a microscope through the screened windows of the can.

A short copper cylinder of about the same circumference as the electroscope disc was provided with an insulating handle, and was connected by a copper wire to one terminal of an electric machine. This cylinder was usually placed upon the glass plate immediately above the electroscope disc, and was then charged by means of the electric machine. Any deflection of the electroscope leaves due to induction by this charge was then observed.

The films were at first deposited upon window glass and were very irregular in thickness and no attempt was made to measure their thickness. When the thinner films were used a very apparent movement of the electroscope leaves was observed. The thicker films were opaque to the induction. One film, in particular, aroused interest. The silver deposit was uneven, being very heavy at one edge and shading off very thin at the opposite edge. When the charged cylinder was placed over the thin part the electroscope was plainly affected, and as it was moved

along toward the thicker edge a position was found where the film was opaque to the induction.

Pieces of plate glass 5 mm. thick and 10 cm. square were then procured and silvered. Great difficulty was experienced in preparing smooth, uniform films thin enough to show induction. After the greatest precautions had been taken in cleaning the plates and in preparing the solutions most of the films would be streaked or spotted. Most of the great number prepared were not worthy of measurement, either on account of irregularities in thickness or because they were not near the limiting thickness which was to be determined.

The thickness of the films was determined by weighing. After the electrical and optical transparency of the film had been observed, its dimensions were measured and the silvered plate was counterbalanced by a similar plate and weighed on a sensitive balance. The silver film was then wiped off by a piece of chamois leather or a cotton towel and the plate was reweighed. A two mg. weight and a one mg. weight were the only pieces ever changed in the weighing.

The optical transparency of the films was determined by measuring the percentage of transmitted light of wave-length about 600μ by means of a Lummer-Brodhun spectral photometer. Readings were taken through various parts of the film and a mean value used.

The following table gives the results of this work in 1907-08. The films numbered 1 to 9, inclusive, were made on glass plates from old photographic plates, the others were made on plate glass. The results are not uniform and there are apparent contradictions, especially in the transparency to light, as in plates 1, 2, and 6. It would seem, however, that at a thickness of about 40 to 50μ the silver film becomes opaque to electrostatic induction, while plates thinner than this allow induction to take place through them.

Plate.	Character.	Wt. in Mg.	Area in cm^2 .	Thickness in μ .	Electric Transparency.	Light Transparency, Per Cent.
1	Uniform	2.7	78	33	Transparent	25
2	Uneven	2.84	82	33	Transparent	53
3	Uniform	6.2	66.5	88	Opaque	3.4
4	Uniform	4.48	77.2	55	Transparent	22
5	Uniform				Opaque	9.4
6	Uniform	5.49	82.8	63	Opaque	33
7	Uniform	3.9	77.9	47	Opaque	17.5
8	Uniform	4.5	78.3	55	Opaque	
9	Uniform	2	78.3	24.3	Transparent	
12	Uniform	2.02	53.6	36	Very trans.	27
13	Very thin				Very trans.	50
14	Uniform	1.9	78.3	23	Opaque	

This investigation was again taken up in 1912 with some variations in the apparatus used. The films were deposited as before upon pieces of plate glass. A wooden box covered with tin foil was used as the hollow conductor. The front of the box was open, the electroscope being shielded by a flap of tin foil hanging from the top about half way down. The lower part was used as a window through which the electroscope could be observed. An aluminium leaf electroscope provided with a scale for reading the deflection of the leaf, and mounted in a tin box with a glass front, was placed in the wooden box with its charging disc centrally below a square opening cut in the top of the box. The top of the electroscope disc was about 5 mm. below the upper surface of the box. The silvered glass plate was then laid with its silver film in contact with the tinfoil over the opening in the top of the box. A copper disc of about the same diameter as the electroscope disc and provided with an insulating handle was used to induce the charge in the electroscope.

The charge was not taken direct from the electric machine, as in the earlier work. Instead, a Leyden jar was charged from the machine and this was connected through a Braun electrometer to the copper disc. By this means, the voltage of the disc could be read up to 3,500 volts.

The thickness of the films was determined as before, a 2 mg. weight being the only one changed during the weighing.

The data determined from these observations are given in the following table.

Plate.	Character.	Wt. in Mg.	Area cm ² .	Thickness in μ .	Voltage of Disc.	Electric Transparency.
I.	Uniform.	4.3	91.9	44.7	3,500+	Opaque.
II	Uniform.	2.1	91	22	3,500	1 scale division.
III.	Fairly uniform.	1.7	64	25	1,600	22 scale division.
IV. ¹	Very thin.	4.03	93	41.3(?)		Very transparent.
V.	Thin, uniform.	1.63	89	17.4	1,000	Very transparent.
VI.	Uniform.	1.62	85.5	18.1		Transparent.
VII.	Uniform.	1.93	92.6	19.8		Transparent.
VIII.	Uniform.	3.36	94	33.7	3,500+	Opaque.

From this series of plates it would seem that the limiting thickness was about 30μ , but the voltages used were much lower than in the former series when the inducing cylinder was connected directly with the electric machine. The results of the two series seemed to indicate, however, that at some limiting thickness of from 30 to 50μ a silver film when used as part of a hollow conductor becomes opaque to electrostatic induction from outside.

¹ No. IV. seems to give discordant results. It looks to be one of the thinnest films made, but its weight was apparently over 4 mg.

During these experiments it was observed that if the electroscope within the hollow conductor was previously charged it became much more sensitive to induction from without. Another important difference was observed between the charged and the uncharged electroscope. If a very thin silver film was placed over the uncharged electroscope and the charged disc was placed upon the glass plate upon which the film was deposited (thus being insulated from the film by the 5 mm. thick glass plate) the leaf of the electroscope would fly out to a large deflection, then would gradually fall to a lower position which it would hold. When the charged disc was removed from the plate the leaf was again deflected and then fell to its zero position. If the electroscope had been previously charged this second deflection was in the opposite direction from the first, indicating that it was due to an inducing charge of the opposite sign.

This seemed to indicate that the film, which was in the beginning transparent to outside induction, became a screen only after a charge opposite in sign to the charge on the copper disc had gathered upon it. When the charged disc was removed this induced charge upon the film then induced an opposite charge upon the electroscope disc which remained until the charge on the film was again distributed over the hollow conductor. That is, the screening effect of the film seemed to depend not upon its thickness, but upon its conductivity.

The first measurement of the specific conductivity of thin films was made by Miss Isabelle Stone in 1898.¹ Miss Stone used silver films prepared in the same manner as those used in this work. She found that for thin films the observed electrical resistance is very much greater than that calculated from their weight, density and dimensions. She found that the ratio of the observed to the calculated resistance was large when the thickness was less than $250\mu\mu$, and was tremendously large when less than about $20\mu\mu$.

Professor J. Patterson² also investigated the conductivity of platinum, bismuth and silver films deposited in vacuo by the cathode discharge. He found that for thin films the specific resistance is several times as great as the specific resistance of the metal from which they are deposited. He was unable to prepare very thin silver films, but he found that for platinum films the limiting thickness at which the specific resistance began to increase greatly was between $7.7\mu\mu$ and $4.6\mu\mu$. Following J. J. Thomson's theory, he regards this change in specific resistance as depending upon the ratio of the thickness of the film to the mean free path of the electrons in the metal.

¹ PHYSICAL REVIEW, Vol. VI., p. 1, 1898.

² Phil. Mag., VI., Vol. IV., p. 652.

Vincent¹ used the resistance of silver films to determine the radius of molecular attraction. He gave as his conclusions, "A film of silver which is greater than $50\mu\mu$ in thickness is composed of a homogeneous layer of constant specific resistance, which is between two layers of less but fixed conductivity whose resistance is constant." He established a formula for calculating the specific resistance of thin metallic films; thus

$$\frac{1}{r} = -A + ce,$$

where r is the resistance, c the specific resistance, which is constant, and e the thickness of the film. This formula seemed to hold for films having a greater thickness than $50\mu\mu$, which thickness he regarded as the sum of the two transition layers on opposite sides of the film, thus making the radius of molecular attraction $25\mu\mu$. Patterson, using Vincent's data in a formula proposed by J. J. Thomson, obtained $60\mu\mu$ as the mean free path of the electrons, or the thickness at which the specific resistance should begin to change.

Moreau² in making investigations on the Hall effect in silver films obtained observed values which were comparable with the theoretical values derived from Vincent's formula, thus confirming Vincent's work. Quincke,³ who made the first direct measurements upon the radius of molecular attraction in silver, found this magnitude to be greater than $50\mu\mu$, while C. W. Chamberlain⁴ using a similar method found a value of $1.5\mu\mu$.

While the estimates of the radius of molecular attraction vary greatly, the investigations upon the conductivity of thin films all indicate a great change in specific resistance at about the thickness where the films used in this investigation became opaque to electrostatic induction. Accordingly, a number of observations were made with plates of poor conducting material placed over the opening in the hollow conductor above the electroscope, and with metal plates insulated by mica or glass and connected to the hollow conductor or to the earth through high resistance. It was found that a glass plate with no silver film on it acted like a very thin film, but much more slowly. The following table gives the electro-scope deflections induced through film V and through a clean glass plate of the same thickness as the one upon which the film was deposited. The figures in the time columns give the time elapsed after the charged disc was placed upon the plate or removed from it. In Fig. 1 of Plot I., the

¹ *Annales de Chimie et de Physique* (7), XIX., 421, 1900.

² *Journal de Physique* (3), X., 478, 1901.

³ *Pogg. Ann.*, Bd. 137, p. 402, 1869.

⁴ *PHYSICAL REVIEW*, XXXI., p. 170, 1910.

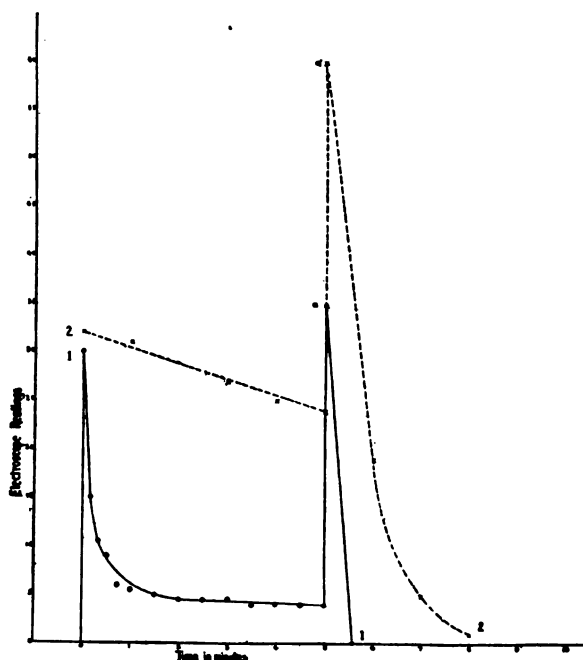


Fig. 1.

Silver Film V ; Thickness $17.4 \mu\mu$.			Glass Plate.		
Time in Sec.	Scale Divisions.	Charge in Volts.	Time in Min.	Scale Divisions.	Charge in Volts.
0	30	1000	0	32	1000
10	15		1	31	980
20	11		2	29	950
30	8		3	27	930
50	7		4	25	910
60	6		5	24	900
90	5	900	Charge removed.		
120	4	880	0	60	950
150	4	840	1	18	
180	4		2	5	
210	3		3	1	
240	3				
270	3				
300	3	800			
Charge removed.					
0	35	860			
30	0				

deflection of the electroscope leaf is plotted against the time elapsed after the charged disc was placed upon the glass plate of the film, and after it was removed. It will be seen from this that the shielding charge

gathered upon the film almost entirely in one minute, after which time the deflection remained constant until the inducing charge was removed. When, after five minutes, the inducing charge was removed, the instantaneous deflection of the electroscope leaf is shown at point *a* on the plot.

Fig. 2 gives a comparison plot for the unsilvered glass plate. In this case the shielding charge gathered much more slowly, so that in five minutes the electroscope deflection fell only from 32 to 24 scale divisions. When the inducing charge was removed, the instantaneous deflection

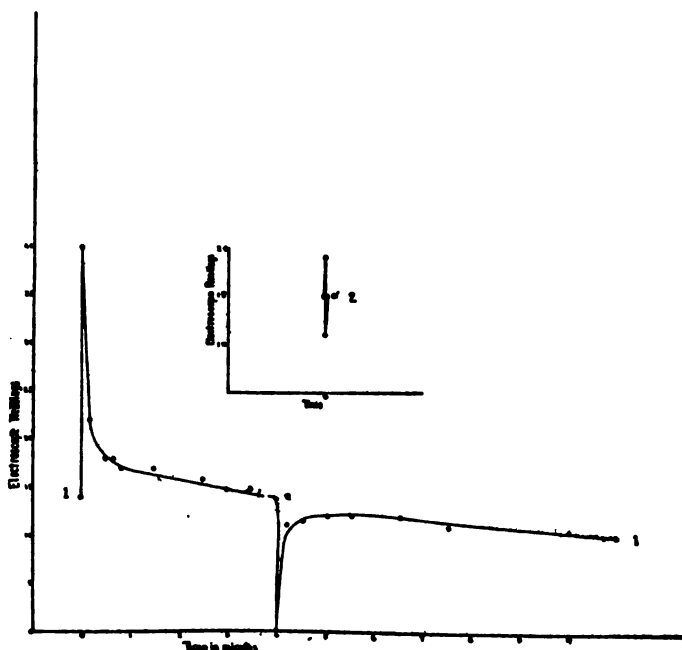


Fig. 2.

was much greater than in the case of the film, showing that the induced charge was dissipated more slowly.

Similar observations were taken for a number of films, but all behaved in the same manner. When the film was insulated from the hollow conductor, either by placing the glass side next to the box, or by placing a sheet of glass or mica between it and the box, the shielding charge did not gather at all, or gathered very slowly. When the inducing charge was placed above the film the electroscope was deflected, and when the inducing charge was removed the electroscope leaf fell to its zero position. This was true whether the electroscope was charged or not. It was equally true when metal plates of different thicknesses were placed over the opening in the hollow conductor and joined to it or to the earth by

a very poor conductor. In this case, the thickness of the metal screen had no appreciable effect, a copper slab about 1 cm. thick acting like a thin film under the same conditions.

A large number of experiments of this kind were made. A hollow conductor was made from a tightly soldered tin can with wire gauze windows. A hole was cut in the top, and the electroscope was placed with its disc just below this opening. A copper plate much larger than the opening in the tin can was laid over this opening and insulated from the can by a very thin disc of mica. When the charged disc was moved toward or away from this plate the electroscope was plainly deflected. When the electroscope was charged, these deflections were in opposite directions when the charged disc was approaching and when it was receding from the plate.

A copper wire was then soldered to the copper plate. When this wire was held in the hand the electroscope was plainly deflected by moving the charged disc above the plate. The same was true when the wire was wound tightly about a gas pipe or about a steam radiator in the room, but when the wire was soldered to the gas pipe no deflection could be observed. The hand could be pressed upon the copper plate and induction would still take place through it; but if one finger were placed upon the plate and another finger of the same hand upon the hollow conductor beside the plate, no induction could be observed. No induction could be observed when the plate was connected to the tin can through soldered contacts and 100,000 ohms resistance, but the induction was large when the two were connected through a hemp string or when the copper plate was insulated by filter paper. If the paper were wet, no induction could be observed. Induction could plainly be observed through a sheet of tinfoil which was spread over the opening in the can and was held down only by its own weight, and it was even observed through a sheet of tinfoil which was soldered to the can at its four corners.

Since the screening effect of the silver films seemed to depend only on their conductivity, and not at all upon their thickness, a thick film which served as a perfect screen when it was laid on the tinfoil covered box, was cut into squares by using a sharp knife and cutting through to the glass along a ruler laid on the film, thus making narrow slits across which the charge would have to leak in order to gather on the film. In this condition it became very transparent, and acted like a very thin film.

A piece of tinfoil pasted on plate glass allowed some induction to take place through it when laid with its foil side on the tinfoil covered box, but this induction was only instantaneous, and the electroscope would

at once return to zero. If the electroscope was charged, a sudden deflection and an instantaneous dropping back to the former position on the scale was observed when the charged disc was placed on the glass above it. When the disc was removed, the deflection was in the opposite direction, and the return to the original position was again instantaneous. The tinfoil was then cut into squares, as in the case of the silver film already mentioned. It now acted like a thin silver film.

Fig. 1, Plot II., shows the electroscope deflection at successive intervals of time after the charged disc was placed above the cut tinfoil, and after it was removed. The electroscope was charged so that its leaf stood at 14 of the scale before the charged disc was brought near. When this disc was placed on the glass above the tinfoil, the leaf was deflected at once to 40. It then fell back, rapidly at first and more slowly later until after 4 minutes it again stood at 14. The charged disc was then suddenly removed. The electroscope leaf at once fell to zero of the scale and then came back rapidly to near 14 again, after which it held its position until its charge leaked off.

Fig. 2 of Plot II. shows the induction through the same tinfoil sheet before it was cut into squares. In this case the deflection each way was much less, and its fall back to its zero position was almost instantaneous.

These experiments seem to make it very probable that electric induction does take place readily through metals until the inducing charge is counterbalanced by a charge of the opposite kind which is induced upon that surface of the metal which is nearest to it. The assumption that the apparent induction through metals is due to a charge like the inducing charge which is driven to that part of the metal screen which is farthest from the inducing charge would seem to be contradicted by the phenomena observed in thin films. In this case, if opposite charges are induced upon the two surfaces of the film, these charges are very close together,—less than $1/10$ a wave-length of light apart, in many cases. The electroscope disc upon which the charge was induced was, in these experiments, about 5 mm. distant from the film, or more than 100,000 times the distance between the two charges. If both could act upon the disc, their mutual effect would be zero. If only the one on the lower side of the film could act, the other being screened by the film, then the film became a screen when this lower charge had leaked away. There would accordingly have been no inducing charge left upon the lower face of the film when the charged disc was removed from above it, and the charge upon the upper surface could have produced no effect upon the electroscope until it had passed through to the inner surface of the film. But according to all theories of hollow conductors, an electric charge placed

upon the exterior surface will not penetrate to the interior surface unless there is a charge of the opposite sign within the hollow conductor.

Also, from J. J. Thomson's theory, since the thickness of these films was much less than the mean free path of the electrons within the metal, the charge could not have gathered upon either surface, but must have been distributed throughout the film. Two opposite charges upon the same film would in such a case be impossible.

This work was carried on in the Physical Laboratory of Stanford University, under the supervision of Professor Sanford.

STANFORD UNIVERSITY,

June, 1912.

NOTES ON THE ELASTIC PECULIARITIES OF PLATINUM-IRIDIUM WIRES.

BY L. P. SIEG.

1. *Introduction.*—The interesting elastic properties of platinum wires alloyed with iridium have been discussed in several papers.¹ The hope has been expressed at several times during the work of determining the equations that would represent the elastic action of these wires. The only matter that seemed to stand in the way of one's obtaining such definite mathematical relations was the lack of a sufficient variety of data. During the past two years many additional observations have been made on these wires, but in spite of the additional information gained by these experiments, no successful mathematical theory has been reached. However, it was thought best to present some of the more interesting elastic peculiarities of these alloys. Many of these results are very suggestive in the general subject of elasticity.

2. *Test of Hooke's Law.*—For many of the phenomena connected with these wires the former papers should be consulted. Only one result will be repeated here, as it is of importance in this section of the work. The wires which are used as the suspensions for torsion pendulums are alloys of platinum with various percentages of iridium. Most of the results are on a forty per cent. alloy, and it will be understood that this particular alloy is meant unless the contrary is stated.

The pendulum is twisted from the zero position to an amplitude of about 12° per cm. of length, and its period is determined as a function of its amplitude.² In Fig. 1 is shown a curve representing a typical results of such an experiment. For the special methods of observation, timing,

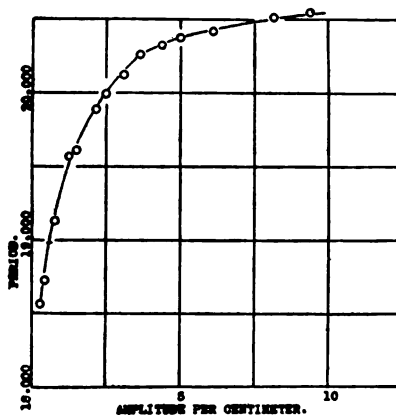
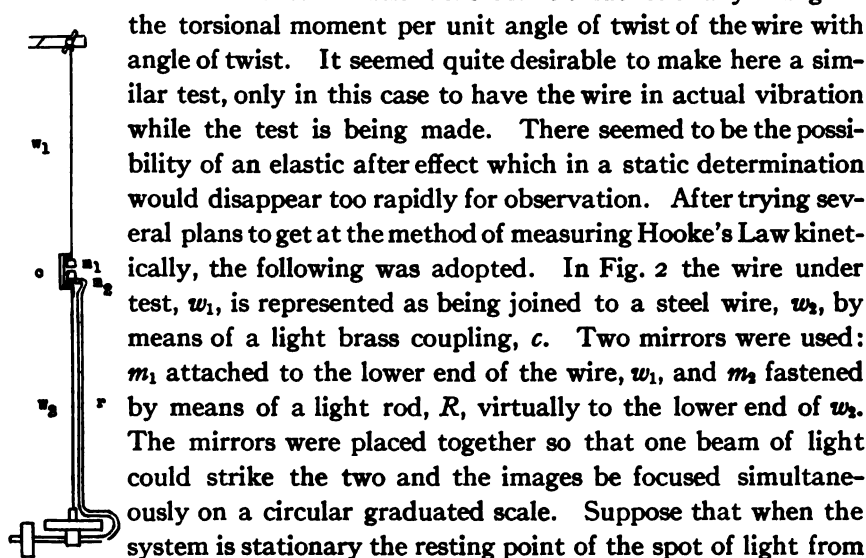


Fig. 1.

¹ For references to former papers see *Phys. Rev.*, XXXI., 1910, p. 421.

² In former papers the word "amplitude" was used to represent the complete range of vibration. In the present paper it is used in its ordinary sense as referring to the angular displacement from the center.

etc., the former papers should be consulted. One of the first thoughts that occur to one in observing this curve is that perhaps here is a case where Hooke's law does not hold. The large period which stays almost constant down to an amplitude of about 5° per cm. of length seems to correspond to a constant and small restoring torque in the wire. Down to this place on the curve the conditions seem to be nearly normal. However, below this amplitude the period falls off rapidly, indicating thereby the possibility of an increase in the restoring couple per degree of twist. Some experiments were made to test this point. In the experiments recorded in some of the former papers¹ static determinations had verified the fact that Hooke's Law holds almost exactly up to the maximum twist. So from static determinations there is no evidence of any change in



the torsional moment per unit angle of twist of the wire with angle of twist. It seemed quite desirable to make here a similar test, only in this case to have the wire in actual vibration while the test is being made. There seemed to be the possibility of an elastic after effect which in a static determination would disappear too rapidly for observation. After trying several plans to get at the method of measuring Hooke's Law kinetically, the following was adopted. In Fig. 2 the wire under test, w_1 , is represented as being joined to a steel wire, w_2 , by means of a light brass coupling, c . Two mirrors were used: m_1 attached to the lower end of the wire, w_1 , and m_2 fastened by means of a light rod, R , virtually to the lower end of w_2 . The mirrors were placed together so that one beam of light could strike the two and the images be focused simultaneously on a circular graduated scale. Suppose that when the system is stationary the resting point of the spot of light from m_1 is represented by x_1 , and of that from m_2 by y_1 . Suppose now the system is twisted. Let the new reading of the spot from m_1 be x_2 , and from m_2 be y_2 . The twist of the wire w_1 is represented by $x_2 - x_1$ while the twist of w_2 is represented by $(y_2 - y_1) - (x_2 - x_1)$. With this apparatus we are prepared to test Hooke's Law both statically and kinetically, provided we assume that the above law holds for the steel wire. This latter assumption seems reasonably justified, at least within the experimental possibilities of this apparatus. For static determinations the lower weight was twisted, after the zero positions of m_1 and m_2 had been determined, was clamped at various degrees of twist, and the several corresponding readings of m_1 and m_2 were determined. In the kinetic observations there were two plans used. In the first place

¹ Loc. cit.

the system was vibrated and the simultaneous turning points of the spots of light from m_1 and m_2 were noted by two observers. These observations gave information of course at the turning points only, and in a way this experiment is not essentially different from the former static ones, for the wires were at the moments of observation, momentarily at rest. In order to test the action during the actual vibration of the wire, recourse was had to photography. The camera was focused on a portion of the scale where previous observation had located the simultaneous appearance of the two spots of light. The rest points of m_1 and m_2 were of course first determined, then during the actual vibration an instantaneous exposure of the two spots of light was made. Exposures were taken at several different parts of the scale, corresponding to various degrees of twist, and of parts of the vibration. The results of these observations, which to save space are not presented here, indicate clearly that within the accuracy of the method used, the restoring torque is at all times, whether the wire is in motion or not, proportional to the angular twist. So one must look elsewhere than to the failure of Hooke's Law for the explanation of the curve in Fig. 1.

3. *Test for Angular Harmonic Motion.*—An indirect test of the question raised in section 2 above seemed possible. That is, if Hooke's Law fails, then perhaps the motion of the vibration is not angular harmonic. It seemed worth while then to test this point. At first an attempt was made to arrange below the wire under test a second torsion pendulum having a steel suspension. The plan was to use two mirrors, similar to the arrangement in Fig. 2, to adjust to equality the periods and amplitudes of the two pendulums, and then to set the two going simultaneously from the same point of rest. It was hoped that if the vibration of the platinum-iridium wire was not truly angular harmonic, that the two spots of light, while starting together, and reaching possibly the extreme amplitudes at the same time, would not at the other points of the scale be together. It was found impossible to adjust the two pendulums together, for while the steel one was practically isochronous, the other one, as indicated in Fig. 1, changed enormously in its period with varying amplitude. The method was then abandoned on account of the many practical difficulties encountered. The next method, and the one that seems to have been successful, was to get a trace of the vibration on photographic paper mounted on the revolving drum of a chronograph. As the width of the chronograph cylinder was only 25 cm. it was necessary to have a small vibration. This in no way caused any difficulty, for the length of the wire was reduced to such an extent that the same twist per unit length as in former experiments was possible, even with the very

limited amplitude. A vertical slit of light was focused on the drum of the chronograph in front of which was a thin horizontal slit. This gave on the drum a very fine point of light, and this arrangement had the additional advantage over the focusing of a point of light in that small vertical oscillations of the vibrating pendulum were not recorded. Thus a perfectly smooth curve was obtained. The room was darkened, the drum covered with the photographic paper, and the pendulum and the drum were started. When all was steady a cardboard stop was removed from the front of the horizontal slit, and light was admitted until about fifteen or twenty semi-vibrations had taken place. The paper was developed, and when carefully dried was ready for measurements. The paper was of an excellent heavy stock and if there was any shrinking it was thought that it would not affect any one measurement more than any others. Several such records were obtained. Now before final measurements on these curves can be made, two reductions must be carried out. The first is the reduction of the tangent of the angle of twist to the angle, for the drum was tangent to the circular scale. The next reduction was a little more difficult to make, and also a little more uncertain. Those who have read the former papers will recall that the logarithmic decrement of the vibration of these wires is not only variable, but is also very large. So the traces on the photographic paper were similar to such a highly damped vibration as that represented in Fig. 3. This figure is, however,

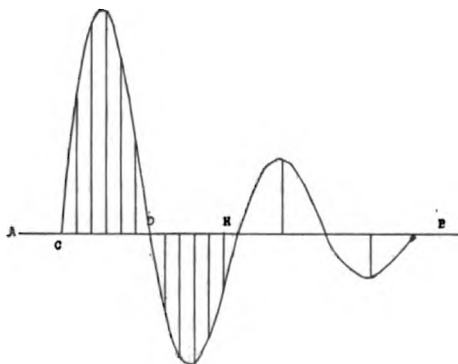


Fig. 3.

much exaggerated. In the first place along the central line AB , each segment such as CD is divided into n equal divisions. In the present work n was made equal to ten. Ordinates were erected at each of these divisions. Let the ordinates erected at the middle points of the segments be represented in length by a_1, a_2, a_3 , etc. The times represented by the distances between these small divisions are each

taken as unity. Then in Fig. 3, the times at which the amplitudes a_1, a_2, a_3 , etc., occur are 3, 9, 15, 21, etc., units. Then the amplitude of any of the median ordinates a_1, a_2, a_3 , etc., is supposed given by the expression

$$a_0 = a_1 e^{mt},$$

where a_1 is any measured median ordinate, a_0 the first measured median

ordinate, t the time elapsed in the arbitrary units, while m in this weir is a variable function of the time elapsed, and not a constant as it is in most wires. The value of m as a function of the vibration number is platted in Fig. 4. After m is determined, any other than a median ordinate is measured and reduced to what it would be were there no damping. To illustrate briefly the general method, a special case from Fig. 3 will be taken. Suppose we take the measurement of the ordinate at H . To get its adjusted length it is adjusted by multiplying by e^{11m} ; the number 11 representing the number of units of time elapsed since the

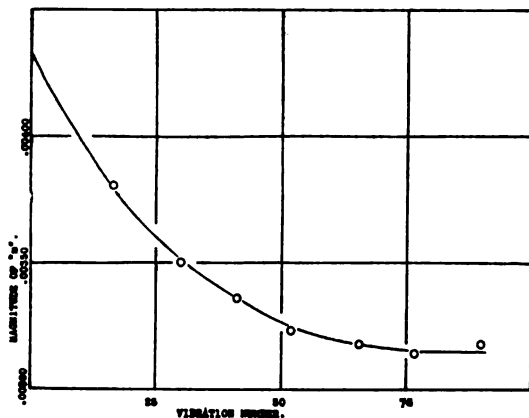


Fig. 4.

beginning. The value of m is taken from the curve in Fig. 4 at the time interval 11. This is found to be about .00410. After all these ordinates are measured and adjusted (and there were over ninety of them to each curve) to what would have been their magnitude had there been no damping, all the corresponding ordinates were averaged. Lastly these were compared with multiples of the natural sines of the corresponding angles. The constant multiplier of $\sin \theta$ was so taken that the values agreed at 90° and of course at 0. The results of a large series of such measurements and reductions are given below in Table I. It seems then as a reasonable result of these measurements and reductions, to assume that the vibrations of the torsion pendulum were angular harmonic. And this in turn gives us additional evidence of the validity here of Hooke's Law during the vibration.

TABLE I.

θ	Const. $\sin \theta$	Means of the Adjusted Ordinates.
0	0	0
15	3.32	$3.29 \pm .01$
30	6.41	$6.39 \pm .02$
45	9.06	$9.03 \pm .02$
60	11.09	$11.08 \pm .02$
75	12.37	$12.35 \pm .01$
90	12.81	$12.81 \pm .01$

It might be mentioned incidentally under this section that from the photographic records there was an opportunity of measuring the period, and amplitude of course, of the vibrating pendulum. These measurements give results identical with those obtained with the method of observation used in the preceding work, and so give one the assurance that the methods there used were sound.

4. *Effect of Long Continued Rest.*—The relation between amplitude and period in these pendulums is not by any means a constant one, as reference to former papers will make plain. Briefly, if the wire has been annealed at red heat, and then vibrated through not more than one or two degrees per cm. of length, its period is low and nearly constant over this range of vibration. However, if the same pendulum is vibrated or even twisted through ten or twelve degrees per unit length, and then vibrated through the same range as before, its period is found to be not only much larger but also to change much more rapidly with amplitude. These two points are shown in curves (e) and (a) respectively of Fig. 5.

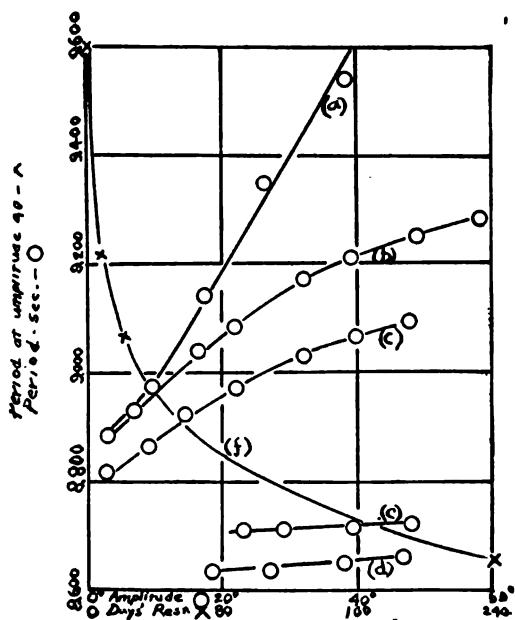


Fig. 5.

In all the experimental work the general method of wiping out the history of the wire was to anneal it under load at red heat. In every case this gave the wire the smallest period. By chance the wire had been left hanging for several months without having been annealed before this long period of rest. On vibrating the wire at the end of this period I was surprised to find from the relation between amplitude and period that the effect of the long rest had been not only similar to the effect of annealing, but that it had actually carried the wire beyond this point. Here was

a case where the wire, as it were, automatically annealed itself. The results of this experiment are shown in curve (d), Fig. 5. To make certain of the matter, the wire was annealed and the old position of the curve for period and amplitude was obtained (curve (e)). Of course this suggested that there was a kind of recovery in the elastic condition

of the wire and to gain information on this point, observations were made after 22 hours' rest (curve (b)), and after 25 days' rest (curve (c)). The original observation had been made after 240 days' rest. In curve (f) of Fig. 5 is represented the relation between the period at a double amplitude of 40° and the number of days that the wire has rested. A general logarithmic recovery curve is clearly indicated.

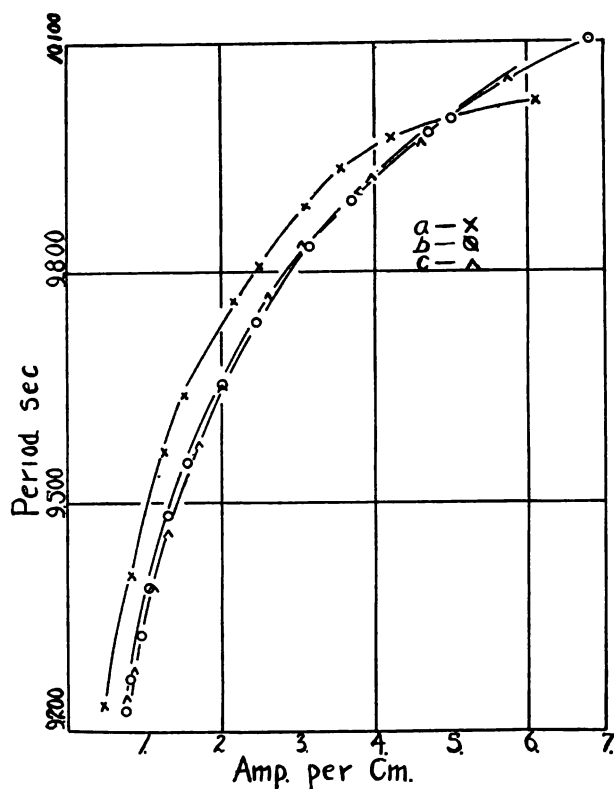


Fig. 6.

5. *Effect of Variation of Length of Wire, and Speed of Vibration.*—In a former paper¹ there were recorded results wherein it appeared to be indicated that even with a careful preliminary treatment of the wire there were differently shaped curves connecting period and amplitude for different periods of vibration. A valid objection to this former work would be that the air friction in the different cases would be so different with different velocities of vibration that the results for internal friction could not well be compared. In view of this a set of observations was made in which external friction was made practically constant. Whether

¹ *Phys. Rev.*, XXXI., 1910, p. 449.

we assume the air friction to be dependent on the first, second, or other power of the velocity of vibration, the air resistance for varying lengths of the wire can be made the same by adjusting the periods so that they are proportional to the maximum amplitude of vibration. Since in the various cases the maximum amplitude of vibration is made proportional to the length of the wire, the period then, simply, is made proportional to the length. This was accomplished without changing the load on the wire, by sliding to or from the center two brass cylinders which fitted over a horizontal steel rod. The results are best explained from the three curves of Fig. 6. These curves are all reduced to the same scale

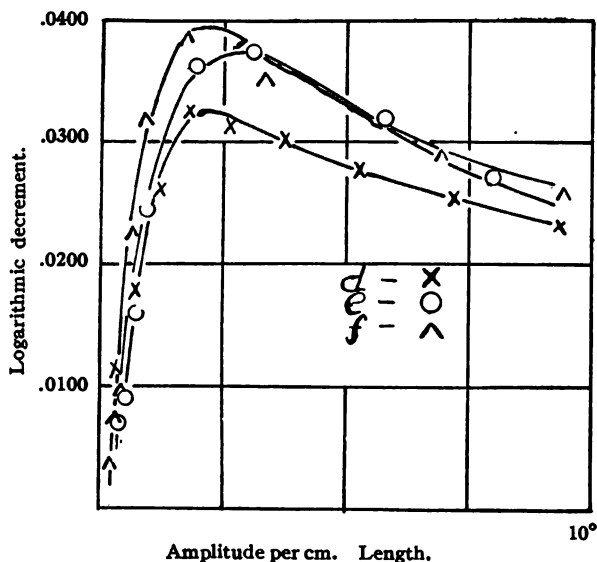


Fig. 7.

so that they can be compared. Curve (a) represents the longest wire, and the longest period of vibration; curve (b) represents the medium, and curve (c) the shortest wire and the shortest period of vibration. The effect noticed in the previous work is here verified. The faster the vibration the sooner the curve connecting period and amplitude turns down toward the X -axis. Of just what significance this is, is at present not clear.

In the above experiments with constant load, with constant initial twist per cm. of length, but with variable length of wire it is interesting to note the variation of the logarithmic decrement with the mean amplitude of vibration in the three cases considered above. These variations are shown in Fig. 7, where curves (d), (e), and (f) refer respectively to the conditions represented by curves (a), (b), and (c) of Fig. 6. Here

again is to be noted a progressive change wherein the shortest wire—likewise the one having the smallest period—has the largest decrement for a given amplitude. The peculiar shape of the logarithmic decrement amplitude curves has been previously noted.¹ It is to be observed that at a twist of from two to three degrees per unit length there is a sudden break, not only in the logarithmic decrement-amplitude curves, but also in the period-amplitude curves, as can be seen in Figs. 7 and 6 respectively.

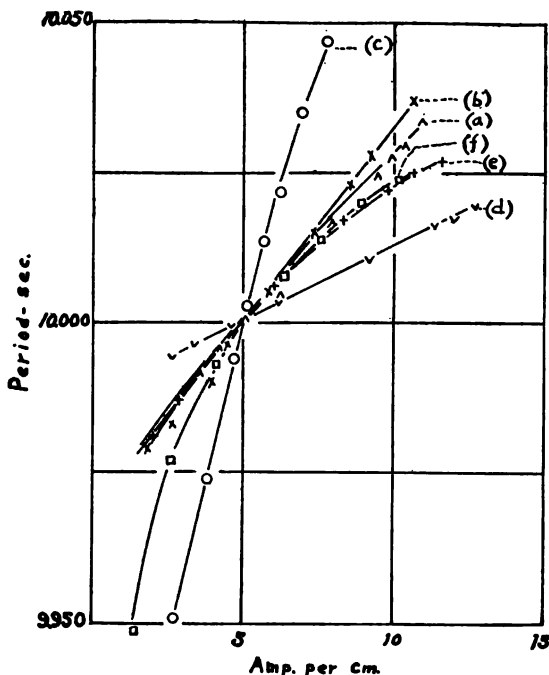


Fig. 8.

6. *Effect of Drawing.*—It was found practically impossible to draw the 40 per cent. wires with any plate we possessed, on account of the extreme brittleness of the wires. Even annealing at red heat had but small beneficial effect. Only with a very gradually graded draw plate could one have much hope of success in drawing down these wires. However, a wire of smaller percentage of iridium, probably 25 per cent., proved to be more easily drawn. The exact composition of this wire is not known. The following procedure was adopted. The wire was annealed and drawn down one size. A piece about 10 cm. long was cut from this, and the remaining part of the wire was again annealed and drawn through the next smaller hole. This was continued until three specimens, represent-

¹ Loc. cit.

ing three successive drawings were at hand. Each specimen was unannealed, but had been annealed just before its drawing. The only difference to be expected from these was that perhaps as the specimens grew smaller there would be a greater fraction of the wire affected by the drawing process. This is of significance in view of the following results. Two sets of experiments were performed. In the first one the three wires were used just as they came from the draw plate, and were vibrated under constant load, nearly constant length, constant initial twist per unit length, and with practically constant period. This latter was accomplished by varying the moment of inertia of the suspended system without changing its mass. In the second set of experiments the wires were first annealed at red heat and subsequently experimented on as in the first set. The results are most easily discussed from the curves of Fig. 8, and from the data in Table II. In Fig. 8 are platted the periods with amplitudes

TABLE II.

No. of Drawing.	Diameter of Wire (mm.).	No. of Vibrations from 12.5° to 4°.	State of Wire.
1	.181	170	unannealed
2	.163	158	unannealed
3	.151	66	unannealed
1	.181	280	annealed
2	.163	170	annealed
3	.151	104	annealed

for these wires. Curves (*a*), (*b*), and (*c*) represent respectively the conditions from the first, second, and third drawings of the wires in their freshly drawn condition, and curves (*d*), (*e*), and (*f*) represent the same wires after having been annealed. Two things will be observed. First the annealing acts to lower the period of a drawn wire just as it does to lower the period of a wire that has been vibrated for some time. Hence drawing and long-continued vibrations have similar effects on the wires. Secondly the effect of drawing becomes more and more pronounced the finer the wires become. This might be expected, for in the smaller wires the ratio of the surface to the volume becomes greater, in fact inversely as the radius, and the drawing apparently has most effect on the surface. In curve (*c*) we have a curve that is almost as pronounced as a similar curve for a 40 per cent. wire. It is evident then that to the quantities to which heretofore have been attributed the relation between period and amplitude, *i. e.*, to the per cent. of iridium, and the previous vibrational history of the wires, must be added the matter of the method used in the drawing of the wires. It will be noticed lastly that not even

an annealing at red heat is able to restore any of the drawn wires to the condition of the wire just larger than itself. Apparently the crystal structure has been so altered by the drawing that heating will not restore it. Probably long-continued rest under load, as described in section 4 of this paper, would restore all the wires to the same condition.

The effect of the drawing and the annealing is strikingly shown in another manner in Table II. In the third column is given the number of vibrations executed by the pendulum in dying down from an amplitude of $12^{\circ}.5$ per cm. to an amplitude of 4° per cm. The two effects mentioned in the above paragraph are here again clearly evident. In each case the annealed wire executes more vibrations over a given range of amplitude than the corresponding unannealed wire, and secondly the finer the wire in each set, the fewer vibrations there are in dying down over the amplitude range. It is certainly striking to have a wire whose cross-section is only two thirds of that of another wire of the same composition, die down in 66 vibrations, while the coarser wire continues to vibrate 170 times over the same range.

7. *Summary.*—Further study has been made of the elastic nature of platinum wires alloyed with iridium. The following points have been made:

(a) In spite of the peculiar variation of period with amplitude, and of the enormous change in the decrement of these wires as they die down in their vibrations, Hooke's Law has been found to hold, not only statically, but also kinetically.

(b) The torsional vibration of these wires has been proved to be practically angular harmonic.

(c) The elastic properties are profoundly modified by long-continued rest under load.

(d) With due allowance for the effects of ordinary external and internal friction, there is found to be a variation in the elastic properties of the wires with the speed of vibration, or at any rate with the period of vibration.

(e) The elastic properties are much modified by drawing the wires.

8. *Conclusion.*—It seems quite definite that all the observed phenomena connected with these alloys must be related to crystalline structure. Several attempts have been made to etch and examine cross-sections of these wires, but thus far no satisfactory specimens have been prepared. If we accept this hypothesis, then it is interesting to note how not only annealing and drawing can change the crystalline structure, but likewise strenuous vibration, and also long-continued rest. It is fortunate that common wires of pure metals and of alloys are not so sensitive to treat-

ment as are these. If they were, we should need in our laboratories the services of an expert historian, if we should ever expect to use the wires in exact measurements. That these effects are present in almost all wires, of course to a much smaller extent than in these platinum-iridium wires, is the belief of the writer, and further with these sensitive wires as exploring agents, we may hope to learn more concerning other elastic substances.

STATE UNIVERSITY OF IOWA,

July 22, 1912.

ANALYSIS OF COMPLEX SOUND WAVES.

BY C. W. HEWLETT.

THE object of this work was the analysis of the complex sound waves sent out by violins. The work is a continuation of that begun in this laboratory by Dr. P. H. Edwards (PHYS. REV., 32, Jan., 1911), who studied quantitatively the analysis of the sound waves sent out from several instruments including the violin, 'cello, flute, cornet and clarinet.

The same method was used by both Dr. Edwards and myself. A set of resonators each provided with a Rayleigh disc was used to analyze the waves. The resonators were cylindrical and were composed of two pieces of telescoping brass tubing, one piece being closed at one end with a solid disc of brass, and the other closed at one end with a disc of brass having a circular opening at its center. This opening was the mouth of the resonator. This arrangement enabled the resonators to be tuned over quite a wide range. A set of thirty-nine resonators was designed with frequencies ranging from 128 to 2,552 vibrations per second. They were constructed by Mr. Frank Smith, one of the mechanics in this laboratory. A drawing showing the design of the resonators, and a table giving their dimensions is given farther on. The resonators were mounted side by side on the four shelves of a wooden case. This was a framework whose front sashes were covered with very thin celluloid, transparent to both light and sound; the other sashes being covered loosely with light tissue paper.

The room in which the experiments were carried out was approximately four meters each way. Reflections from the walls, floor and ceiling distort the sound waves to such an extent that absolute measurements of the intensity of a source would be almost impossible if some means were not devised to eliminate these disturbances. With this end in view the walls and ceiling were hung with a preparation of flax called Linofelt. This material is a fluffy, fibrous substance sewed between two strips of heavy paper. This was first suspended by wires so as to completely hide the ceiling and walls and then the strips were sewed together. The surface of the paper on the walls exposed to the room was then stripped off, leaving the soft hairy surface exposed. On account of the lack of mechanical strength of the material the paper was not stripped from the ceiling. The floor was also covered with Linofelt and then

with a layer of burlap. The reflections were greatly decreased, although not entirely destroyed.

The resonator system for measuring the intensity of sound was as follows: A thin mica disc whose diameter was about half that of the opening in the resonator was suspended by means of a fine quartz fiber close in the opening. A small glass rod attached to the lower side of the disc terminated in a small glass dumbbell which dipped into a cup of kerosene oil. The object of the dumbbell and oil was to damp the motions of the disc. In the center of the mica disc was fastened a small chip of platinized concave mirror. The disc was turned so that its plane made an angle of 45° approximately with the mouth of the resonator.

When the resonator responds to a tone of its frequency the disc experiences a couple whose magnitude depends upon the angle between the plane of the disc and that of the mouth. The disc will therefore turn till the couple due to the fiber is equal and opposite to that caused by the vibrating air. The formula given by Lord Rayleigh for this couple (Theory of Sound, Vol. II.) is

$$M = \frac{1}{8} \rho a^3 W^2 \sin 2\theta,$$

where a is the radius of the disc, W is the mean velocity of the vibrating particles of air, ρ is the density of the air, and θ is the inclination of the disc to the plane of the mouth. Now the couple may be calculated from a knowledge of the torsional moment of the fiber and the deflection, so that the mean kinetic energy per cubic centimeter of the air in the mouth of the resonator, $\frac{1}{2} \rho W^2$, may be calculated. Helmholtz has shown that the intensity of the waves which would exist at a given point were the resonator not there is to be obtained from that in the mouth of the resonator by multiplying by the factor $2R^3/\pi^3 S$, where R is the radius of the mouth and S is the volume of the resonator.

If δ is the deflection in degrees of the disc and if K is the torsional moment of its suspending fiber, then the moment acting on the disc due to the fiber is $2\pi\delta K/360$. When equilibrium has been established, the moment due to the vibrating air and that due to the fiber must be equal and opposite. Equating the two we have

$$\frac{2\pi\delta}{360} K = \frac{1}{8} \rho a^3 W^2 \sin 2\theta,$$

$$\frac{2\delta}{\sin 2\theta} = \frac{480}{\pi} \cdot \frac{a^3}{K} \cdot \rho W^2,$$

$$\text{Kinetic energy per cm}^3 = \frac{1}{2} \rho W^2 = \frac{\pi}{960} \cdot \frac{K}{a^3} \cdot \frac{2\delta}{\sin 2\theta}.$$

The kinetic energy per cubic centimeter in the air is the quantity we wish to determine when analyzing a complex wave. From the above formula it is seen that the kinetic energy per cubic centimeter in the air in the mouth of the resonator may be determined from a knowledge of δ , θ , and the ratio K/a^3 . By using Helmholtz's formula for the multiplying power of a resonator the kinetic energy per cubic centimeter which would exist at that point if the resonator were not there may be calculated. It is thus seen that it is not necessary to know the torsional moment of the fiber and the radius of the disc separately. The method by which K/a^3 was determined will be discussed shortly.

A large translucent paper scale with vertical rulings 1 cm. apart was mounted in front of the case of resonators. The faces of the resonators were adjusted parallel to this screen. Four Nernst glowers, one for each shelf of the resonators, were provided with hoods each containing a small aperture. These were mounted at the sides of the case, the apertures facing small celluloid windows. The rays of light entering these windows were therefore perpendicular to the axis of the resonator.

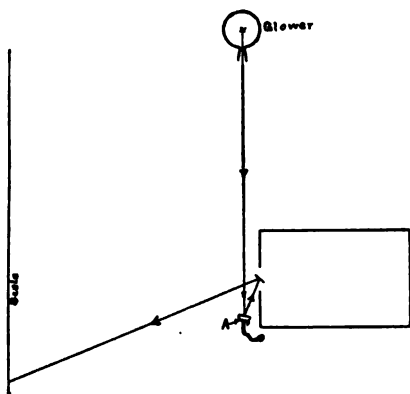


Fig. 1.

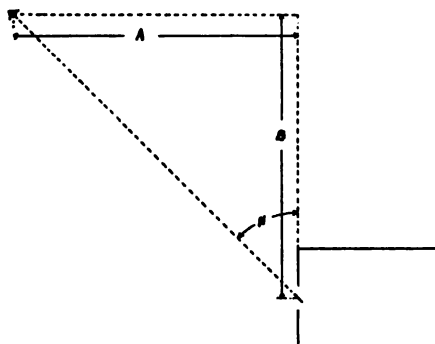


Fig. 2.

The disc was turned so that the chip of the mirror on its face was hidden from the glow and an auxiliary mirror A mounted on a lead wire served to illuminate its surface. By this means the image of the glow on the scale could be placed where desired without interfering with the disc or glow. The surface of the mica disc was blackened with soot to avoid reflections from its surface. The deflections on the scale were reduced to degrees by a graphical method. The angle between the disc and the face of the mouth of the resonator was found by placing the eye in the plane of the disc and then measuring the tangent of the angle. The zero of the image was noted at the same time.

$$\tan N = \frac{A}{B}.$$

This angle, of course, was the zero angle, but knowing the deflection, half its value could be subtracted from the zero angle thus giving the θ in Rayleigh's formula.

Edwards showed that, if the disc was very close in the opening of the resonator, the deflection produced by a sound of a given intensity was independent of the distance of the disc from the mouth. In the present work this experiment was repeated and found to agree with the work of Edwards when the size of the disc was small compared to that of the opening; but, if the disc was larger than half the opening, the deflection increased very rapidly with approach to the opening when this distance was small. It was found that with a disc whose diameter was half that of the opening the deflection was practically constant within a distance of about .1 of the diameter of the opening. All the suspensions hung within this distance, and those in front of the larger resonators were much closer.

The torsional moments K of three of the fibers were determined by measuring the periods of vibration in vacuum, first when loaded with the usual system of disc and glass damper, second with a small auxiliary load whose moment of inertia was known. The diameter of the disc was also measured and K/a^3 calculated for each of these three suspensions. K/a^3 for each of the suspensions used in the experiments was then determined by comparison with these standards. To make this comparison a suspension and one of the standard suspensions whose disc was nearest the size of the one which it was wished to compare were suspended successively in front of the same resonator at the same distance from its face. The resonator was excited by an organ pipe or another resonator blown by air kept at a constant pressure. We have from the equation previously given,

$$\frac{K}{a^3} = C \frac{\sin 2\theta}{2\delta}$$

where

$$C = \frac{480}{\pi} \rho W^2.$$

Using the standard suspension, C may be determined. The suspension for which K/a^3 was required was then substituted for the standard suspension and the deflection in degrees of the disc determined. θ the angle between the disc and the mouth of the resonator was also determined. Substituting these quantities in the previous equation K/a^3 was

calculated. K/a^3 was measured for all the suspensions at least three times, and the suspensions were compared among each other in order to get some idea as to the accuracy attainable. The results which are given further on are the averages of all the determinations and are probably correct to within 10 per cent. The comparison resonator was mounted at the center of curvature of a circular paper scale free to rotate about a perpendicular axis through its center of curvature. This scale was divided into degrees, so that the deflections were read directly in degrees. In this case the zero angles were determined by measuring the angle between the image when the disc lay flat against a piece of glass clapped over the opening of the resonator, and when hanging free.

A word should be said in regard to the way in which the resonators were tuned. A violin was first tuned by means of an "A" tuning fork, and then the resonators were tuned to the partials of the violin. The resonator was first tuned approximately by ear. It was then placed in its position back of its disc and its pitch found by playing the violin and sliding the finger along the string till a maximum deflection was obtained. The resonator was then sharpened or flattened as required by sliding the telescoping tubes, and the process repeated until the resonator was so exactly in tune that no difference could be observed between its pitch and the corresponding partial. The pitches of some of the partials of the different strings so nearly coincided that it was found possible in some cases to use the same resonator for a partial of two or three strings.

This work has been confined almost exclusively to the analysis of the quality of the tone from the open strings of violins, although when the apparatus was designed it was intended to include some other instruments. In making the tests the performer took his position in front of the case of resonators so as to hold the violin about the same distance from all the resonators. This distance was about 240 cm. The observer's position was in front of the scale but slightly to one side so as not to obstruct the waves on their way to the resonators. In taking a record, the performer drew the bow to and fro over the open string as evenly as possible several times in succession. On account of the varying intensity from one end of the stroke to the other, and the changes in quality as the position of the bow changed the positions of the spots of light on the scale were quite variable during this interval. However after some practice it was found possible to select a short interval during which the quality of the tone seemed the best and to locate the image with some accuracy. Thus during an interval of from five to ten drawings of the bow to and fro it was usually possible to form a pretty fair estimate of the

deflection for any one partial. This same process was repeated for each partial. The image giving the largest deflection was observed during the whole procedure, and it was ascertained that it kept on the average the same position throughout the test. Should its average position change owing to change of pitch or intensity the performer was immediately notified. It might be mentioned here, that invariably it was found comparatively easy to produce a tone of fairly constant intensity throughout the stroke on the best violins, while with violins of poor quality this was very difficult to do.

Constants for Fibers Used.

Numbers correspond to number of resonator with which the fibers were used.

No.	$K/a^3 \times 10^3$	No.	$K/a^3 \times 10^3$	No.	$K/a^3 \times 10^3$	No.	$K/a^3 \times 10^3$
1	3.4	11	21.1	21	1.93	31	4.34
2	12.3	12	19.7	22	135.0	32	8.0
3	2.0	13	69.0	23	2.23	33	3.88
4	24.8	14	18.9	24	62.5	34	21.30
5	6.9	15	36.5	25	80.4	35	2.97
6	6.5	16	3.91	26	4.59	36	4.52
7	4.8	17	125.0	27	7.61	37	1.08
8	15.0	18	21.4	28	61.30	38	8.74
9	6.4	19	46.3	29	1.61	39	3.00
10	274.0	20	36.4	30	.60		

K represents the torsional moment of the fiber and a the radius of the disc to which it is attached.

Number of Resonator System.	G-string.	Partials.		
		D-string.	A-string.	E-string.
5	1			
8		1		
11	2			
12			1	
15	3	2		
16				1
19	4			
21		3	2	
23	5			
26	6	4		
28			3	2
29	7			
30		5		
31	8			
33	9	6	4	
35	10			3
36		7		
37	11		5	
38	12	8		
39	13	9	6	4

In this investigation only certain of the above set of resonators and fibers were used. These are given on the following page, and it will be noted none of the suspensions for which K/a^3 is very large were used. The following are the partials of the open strings that were investigated. The number of the resonator system used with each partial is also given.

RESULTS.

Below is a curve showing the necessity of keeping the size of the disc used well under the size of the opening in the resonator. In order to make the curves comparable, the deflections were expressed as percentages of the deflection which would exist if the suspension were placed in the mouth of the resonator, *i. e.*, at 0 distance from the mouth. This deflection was obtained by extrapolation.

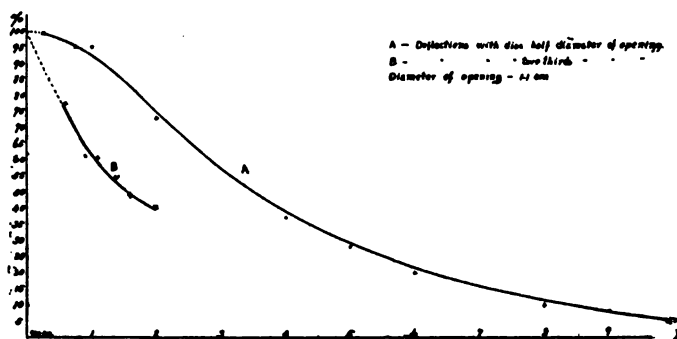
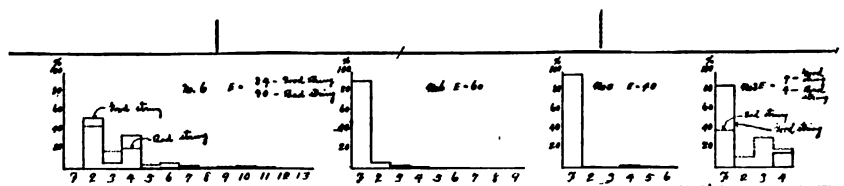


Fig. 3.

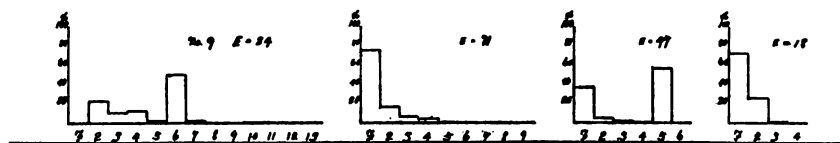
Records were made of the tone quality of twenty-nine violins and one viola. The records of a number of these instruments were made but once, while anywhere from two to six records of some of the others were made. A few of these records illustrating the points brought out in this work are reproduced at the end of this paper. It will be observed that each record is distinguished by a number. These numbers indicate approximately the order in which the records were made. The abscissæ represent the fundamental and successive partials of the string being played. The ordinate at any partial gives the energy in that partial expressed as a percentage of the total energy given out by the string. The energy per cubic centimeter, E , given out by each string is also given. To reduce to ergs per cubic centimeter multiply by 10^{-7} . The distance of the resonators from the position occupied by the violin when the record was being made was about 240 centimeters, so that the output of the violin may be calculated if one neglects the distortion of the waves due to reflections at the walls, and if it be assumed that the violin sends

out waves equally in all directions. The four curves in each set correspond to the open G, D, A, and E strings respectively, reading from left to right, unless otherwise specified.

There are differences in the quality of tone produced by different performers on the same instrument, sometimes even by the same performer at different times. However, fairly good agreement was usually obtained between the records made by the same performer on the same instrument under the same conditions. Some idea as to the degree in which the experiments could be repeated may be obtained by comparing the following two sets of records, (16, 18, 20), (24, 31, 37). The last group were all made by different performers playing with quite different degrees of loudness. The following groups of records are representative of some of the best instruments studied: (56, 57), 19, (16, 18, 20). Record 17 and the group (24, 31, 37) were made from two violins of exceedingly poor tone quality. The violins studied were roughly classified under three heads, the criterion being the general opinion of musical people in regard to the quality of the instrument. About one sixth of the instruments belonged to the class of the "Best Violins," about one sixth to the class of the "Poor Violins," and the remaining two thirds to the class of the "Ordinary Violins."



RECORDS 3 AND 6. These are the records of two violins bought in the white and finished and graduated by Mr. Della Torre. They have been treated with Mr. Della Torre's varnish and oil preparation. Both violins were made as much alike as possible. Record 3 is of the violin designated as "A," while record 6 is of the violin designated "W." Mr. Della Torre played for both records. It was found that both violins were beginning to "open up," i. e., the glue holding the ribs, back and belly together had turned loose in one place. This may account for the fact that the tone was no better, as Dr. Edwards recorded the violins as having very good tone quality.

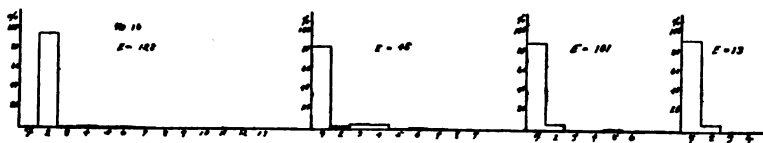


RECORD 9. This is a trade violin bought from Eisenbrandt, of Baltimore, by Mr. Della Torre for \$5.00. It is designated by Mr. Della Torre as No. 2. This record, played by Mr. Della Torre, shows the violin as having a very poor weak tone. The sound post was directly under the bridge, and the base bar quite stiff.

DISCUSSION OF RECORDS.

G-string.—The intensity of the fundamental of the G-string on all the violins is negligible in comparison to the second partial (first overtone). On all the best instruments the intensity of the second partial lies between 90 and 100 per cent. of the total intensity of the string. The third partial has an intensity of less than 10 per cent. of the total intensity of the string. After the third, the intensity of the partials drop off very rapidly, the fourth having less than 2 per cent. of the total intensity, there being no measurable energy beyond the sixth partial.

In the case of the poor violins, the second partial rarely has more than 60 per cent. of the total intensity and very often less than 20 to 30 per cent. The third partial is usually the strongest and attains at times more than 70 per cent. of the total intensity. In one case it was observed that the sixth partial contained 50 per cent. of the total intensity (see record 9). After the third partial the intensity is distributed among the partials as far out as the thirteenth, dying out very slowly in some cases. Nothing more general can be said in regard to the ordinary violins than that the second partial usually contains from 60 to 90 per cent. of the total intensity, the third and fourth partials being often quite prominent. Sometimes the third partial is the most prominent, but in this case it is probable that the merits of one of its other strings saves it.

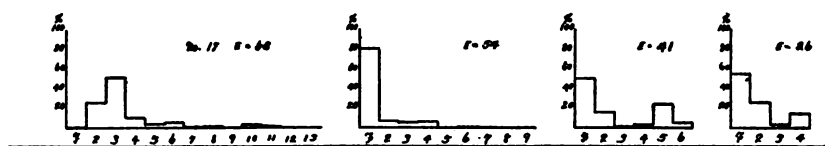


RECORD 16. This violin was made by Luther Heigl in 1911 and is numbered 106. The instrument has a fine tone quality, especially on the G string. Mr. Della Torre played for this record just before applying to it his oil preparation. The intensity of the playing was medium.

D-string.—On all the best violins the intensity of the fundamental on the D-string lies between 80 and 100 per cent. of the total intensity. The remaining 20 per cent. is distributed irregularly between the second, third, and fourth partials, the fifth and sixth receiving some in some cases, but there is nothing beyond the sixth or seventh. On the poor violins the intensity of the fundamental may rise as high as 70 per cent. but usually is as low as 60 per cent. or lower. In one case it was found to be as low as 20 per cent., the second partial being 74 per cent. A peculiarity of these poor violins is the way in which the intensities of the successive partials fall off, "step" fashion (see records 24, 31, 37, 9). On the ordinary violins the fundamental of the D-string may have

intensities as low as 50 per cent. and as high as the best violins. Usually the intensity of the fundamental lies above 70 per cent. and very often the record appears very much the same as do those on the best violins.

A-string.—On the best instruments the intensity of the fundamental lies between 91 and 100 per cent. of total intensity. The intensity of the second partial is less than 8 per cent. while the third and fourth partials are absent. The fifth partial appears quite frequently in all classes of violins. In all the best violins it has an intensity of less than 6 per cent., and there is no energy in the sixth or higher partials. The intensity of the fundamental of the A-string on the poor violins may be anywhere from 30 per cent. to 90 per cent. or more of the total intensity.

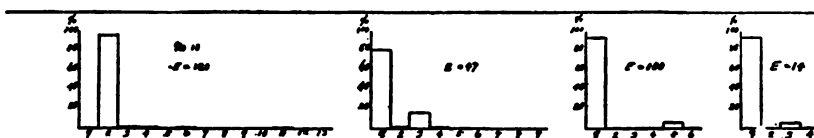


RECORD 17. This violin is owned by Mr. Grubmeyer, of Baltimore. It is a very poor "trade" violin, and has a tone which is very weak in intensity and poor in quality. Mr. Della Torre played for the record.

The intensity of the fundamental usually lies between 50 per cent. and 70 per cent. of the total intensity. The balance of the energy is irregularly distributed among the second, third, fourth, fifth and sixth overtones, the fifth overtone sometimes getting a large share (see record 9). The intensity of the fundamental on the A-string of the ordinary violins may lie between 70 per cent. and 95 per cent. of the total intensity, but most often lies between 80 per cent. and 95 per cent. Some of them resemble the poor instruments in that the energy is scattered among the partials, while most of them approach the records given by the best instruments.

E-string.—In the best violins the fundamental has an intensity of from 70 per cent. to 95 per cent. the total intensity. The balance of the energy appears quite irregularly distributed among the second, third, and fourth partials. It is hardly possible to make a distinction between the ordinary violins as a whole and the poor violins. In both classes the intensity of the fundamental is frequently between 70 per cent. and 95 per cent. Often however it falls below this, and occasionally the second, third, or fourth partial appears the most prominent.

All of the best violins have certain strings which are considered better than others. I have selected these best strings and plotted them on a single sheet which I have labelled "Ideal violin tone." The curves for all the best strings on the best violins fall within the shaded portions of these curves.

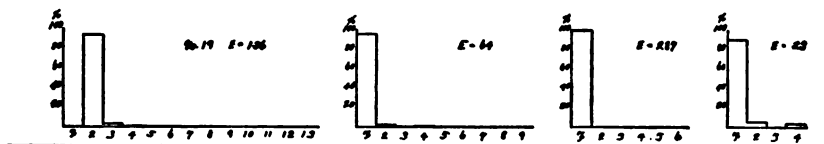


RECORD 18. Same violin as in Record 16. Mr. Della Torre played for this record two days later than for record 16, having applied the oil preparation in the meantime. The "A" string seemed to give a brighter tone than in the previous test. The intensity of the playing was moderate.

The effects of certain changes in the adjustment of the violin were also studied. It is seen that bad strings throw the energy out into the higher partials (see records 3 and 6).

It was thought that strings stopped by holding them down to the finger board with the fingers might give a tone in which the overtones were less prominent than with the open string. This was tried by playing the notes corresponding to the open D, A, and E strings on the G, D and A strings respectively. A comparison of records 20 and 57 will show, however, that while this may be true in some cases, it is not necessarily always true.

The effect of hard and soft bowing was tried at various times. Record 55 shows the results of forcing a very fine violin. Contrary to expectation the overtones are fewer and less intense than with the same instrument played lightly (see record 43). Record 20 shows the result of soft bowing on a good violin. Records 16 and 18 are for the same instrument when bowed with medium intensity. Again it does not appear that on the whole the number and relative intensities of the overtones are decreased. Record 31 represents a violin bowed with medium intensity, and 37 represents the same violin bowed softer.

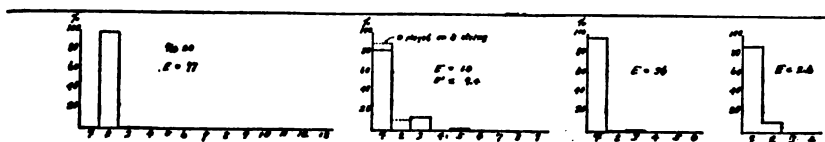


RECORD 19. This violin is owned by Professor Aperda, of Baltimore, the age and make of the instrument are unknown. The instrument has a very fine tone quality, and an exceptionally powerful "A" string. Mr. Della Torre played for the record.

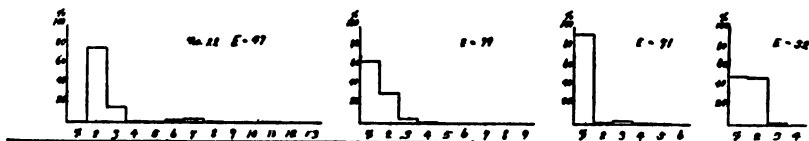
A violin was found in which the sound post was directly under the bridge. A record was made (9), and then the sound post was moved to its proper position back of the bridge. Another record (22) was then made. The total intensity of the sound was increased about 36 per cent. by this procedure, and a wonderful change produced in the relative intensities of the overtones, shifting the energy into the overtones of lower

frequency. The base bar of this same violin was found to be very thick. Two strips were accordingly removed, and a record made after each removal. Finally, the base bar was about half as thick as previously, the intensity about 50 per cent. greater than before the base bar was split and the overtones fewer and the energy more concentrated into the lower overtones (see record 54).

The reflections from the walls is one of the main difficulties in the work, and I would suggest that any one wishing to continue it devise some scheme to do away with walls or put them at a sufficiently great distance.



RECORD 20. Same violin as in records 16 and 18. Mr. Della Torre played for this record on the same date as for record 18. The playing was soft. The D played on the G string seems to be a purer tone than when played on the open strings.



RECORD 22. Same violin as in record 9. The sound post had just been moved to its proper position. It will be seen that the total intensity of the sound was 36 per cent. higher than the previous one. Mr. Della Torre played for the record.

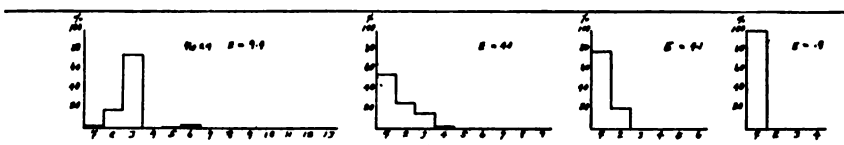
It has been suggested to do the work in a very large tent. I think that the work should be carried on in some locality where the evenings are calm. A collapsible covering should be built around the apparatus in a large field. When one wishes to work the covering could be removed.

SUMMARY.

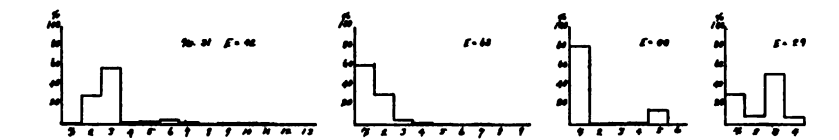
1. Trustworthy records of the tone quality of several violins have been obtained.
2. A set of curves has been drawn defining tentatively an ideal violin tone, the data being taken from the best violins obtainable.
3. Soft bowing does not necessarily produce a purer tone than medium bowing, and hard bowing does not necessarily produce a tone less pure than obtained with medium bowing.
4. The method furnishes a powerful means of studying the effects produced by different kinds of strings and bridges, and by various changes produced in the adjustment of the sound post, base bar and bridge.

In conclusion I wish to thank Professor Ames, Dr. Anderson, Dr. Pfund

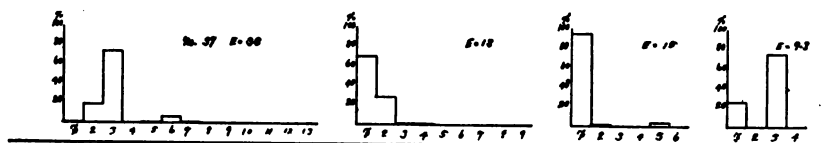
and others in the laboratory for their assistance and advice, and for the interest they have shown in my work. I wish also to thank numerous friends in the city and members of the Peabody Conservatory of Music for the loan of violins and for personal assistance in the work. In conducting this investigation I have had the great benefit of the advice and assistance of Mr. Frank Della Torre, of Baltimore. He placed at my disposal his entire collection of violins, which is a very fine one; he took apart and altered one of these violins for me, a kind of work in which he is exceptionally skillful; and he played various violins for me while I made the observations.



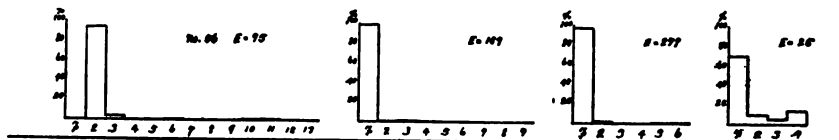
RECORD 24. This violin is owned by Mr. Wear, of Baltimore. It is a comparatively new and cheap instrument, with a weak, scratchy tone quality. The sound post was in front of the bridge. The writer played for this record, and having to make observations at the same time, he was located unsymmetrically with respect to the resonators.



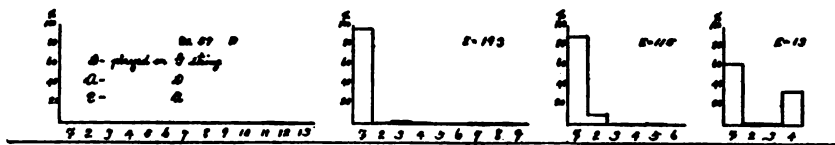
RECORD 31. Same violin as in record 24. Mr. Della Torre played for this record.



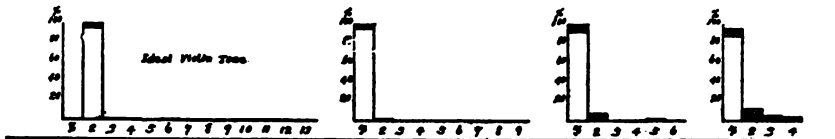
RECORD 37. Mr. Wear played for this record. He played more softly than did Mr. Della Torre in record as above.



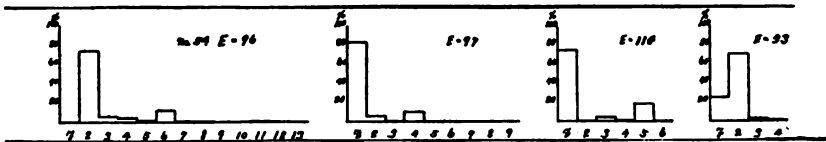
RECORD 56. Professor Van Hulsteyn's violin. This violin is a Regiera and has a very fine tone quality. The "E" string is the poorest string on the instrument. Professor Van Hulsteyn played for the record. The tone was pleasing, and the playing of medium intensity.



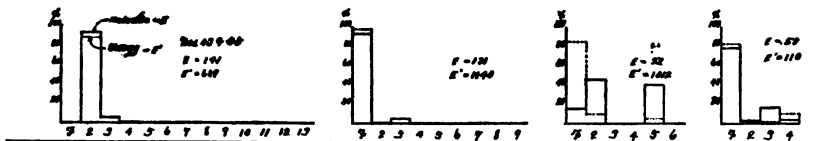
RECORD 57. Professor Van Hulsteyn's violin. The record consists of D played on the G string, A played on the D string and E played on the A string. Professor Van Hulsteyn played for the record, and the playing was of medium intensity.



Ideal violin tone. This set of curves represents tentatively the characteristics of the ideal violin tone. The characteristics of the best strings on the best violins tested fall within the shaded portions of these curves.



RECORD 54. Same violin as in record 9. Two strips had been removed from the base bar, so that the bar was about one half of its original width. The total intensity was about 100 per cent. greater than in record 9. Mr. Della Torre played for the record.



RECORDS 43 AND 55. Grancino violin (1727). This instrument has a powerful tone and a fine tone quality. Record 43 was made with Mr. Della Torre playing. The instrument was played lightly because louder playing sent the spots of light off the scale. Mr. Della Torre also played for record 55. The intensity of the sound was very great. An auxiliary scale was necessary to read the deflections. A bamboo was used. The tone quality of the instrument was very similar to that of Professor Van Hulsteyn's violin.

THE JOHNS HOPKINS UNIVERSITY,
June 4, 1912.

A NEW METHOD OF PHOTOGRAPHING SOUND WAVES.

BY ARTHUR L. FOLEY AND WILMER H. SOUDER.

AT the Chicago meeting of the American Physical Society in 1905, the senior author of this paper exhibited photographs of interference and diffraction fringes about electric discharges and fluid streams, taken by what might be called the point source shadow method.¹ The light from a point source is allowed to fall directly upon a photographic dry plate several meters distant from the source. About half way between the source and the plate is placed the fluid stream or whatever is to produce a shadow on the plate. Say the object is a stream of warm air issuing from a glass tube. The air stream maintains for a time a more or less definite lateral surface and produces diffraction fringes on the plate. The light is refracted on passing through the stream of air of density differing from the air about it, and so the stream casts a sort of shadow on the plate. The method is so sensitive that it gives a shadow of a stream of water in water. Inasmuch as a sound wave in air consists of a compression and rarefaction, the method should enable one to photograph the shadow of a wave, provided a sufficiently strong instantaneous point source of light is obtainable. Mr. Wilmer H. Souder, while teaching fellow at Indiana University, undertook the problem and succeeded in obtaining sound wave photographs, but discontinued the work to accept a position elsewhere. The writer continued and extended the work, and succeeded in obtaining a light source many times as intense as the source previously used—so intense that, in photographing sound waves, no further increase is desirable.

The general arrangement of the apparatus in this experiment is shown in Fig. 1.

The sound wave is produced by an electric spark at the spark gap *S*, which will be called the sound gap. The light is produced by a second spark at the illuminator gap *I*, which will be called the light gap. If the sparks at *I* and *S* are simultaneous the light from *I* passes *S* before the sound wave emerges from behind the spark terminals. If the time interval between the sparks is any considerable fraction of a second the

¹ "Diffraction Fringes from Electric Discharges and from Fluid Streams," A. L. Foley and J. H. Haseman, *PHYSICAL REVIEW*, Vol. 20, 1905, page 399; *Proceedings Indiana Academy of Science*, 1904, p. 206.

sound wave at S passes out of the field before the light from I reaches S . But when the interval between the two sparks is properly timed the sound wave at S casts its shadow on the photographic dry plate P .

The spark gaps S and I are connected in series to the spark knobs K , K_0 which receive sparks from T , T_0 , the terminals of a large electric induction machine having four rotating mica plates thirty inches in diameter, driven by a variable speed electric motor. The length of the spark gaps $T-K$ and T_0-K_0 is adjusted by sliding T and T_0 horizontally, or by moving K and K_0 vertically. Glass plates G and G_0 about twenty centimeters square are fastened to a wooden bar or rod R ,

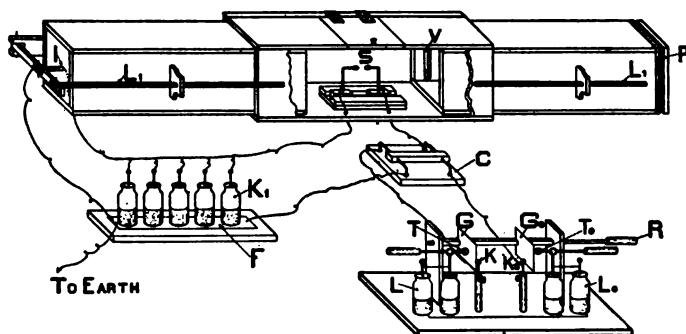


Fig. 1.

arranged so that it can be rotated on a horizontal axis. In the position shown in the figure the plates are directly between the spark knobs T and K , and T_0 and K_0 , thus preventing a spark. A quarter turn of the handle of the rod R removes the plates and allows the spark to pass. In practice, however, it was found best to make the spark gaps $T-K$ and T_0-K_0 just long enough to prevent sparking when the glass plates are out of the field. Rotating the rod then causes a spark just as the edges of the plates pass through the gaps.

To get a strong sound wave at S and an intense light at I , the sparks to the knobs K , K_0 should be heavy, consequently from one to three Leyden jars L , L_0 are put in multiple with each of the jars on the machine itself. Most of the work was done with one additional jar connected to each of the machine jars.

The knobs K , K_0 are connected to the gaps S and I through a large commutator C . Thus the direction of the spark at the gaps is readily changed. It was found that the apparatus worked about equally well with the spark in either direction. The chief effect of reversing the spark direction appeared to be a slight change in the time intervals of the sound and light sparks.

The sound gap S is in series with the light gap I , nevertheless the spark at S occurs before the one at I , because of the capacity K_1 which is in multiple with the gap I , as shown in the figure. The capacity K_1 consists of from two to eight Leyden jars of the usual type, standing on a sheet of tinfoil F , with one coat connected to earth. The time interval between the sound and light sparks depends on the capacity K_1 , the greater the capacity the greater the interval. However it is not at all necessary to have here a variable condenser or one made up of small units. One can vary the capacity K_1 between rather large limits and still keep the proper time interval by varying the circuit in other ways. The time interval is increased by decreasing the capacity L , L_0 , by decreasing the length of the spark gaps $T-K$ and T_0-K_0 , by decreasing the length of the sound gap S , and by increasing the length of the light gap I , the last having the greatest effect. One can vary the capacity K_1 by a hundred per cent. and readily maintain an approximately constant time interval between the sound and light sparks by merely changing the length of the light gap. The length of this gap is adjusted by means of the lever arrangement L_1 , L_2 , as will be understood by reference to both Figs. 1 and 2.

Fig. 2 is a cross section of the light gap, the spark taking place between

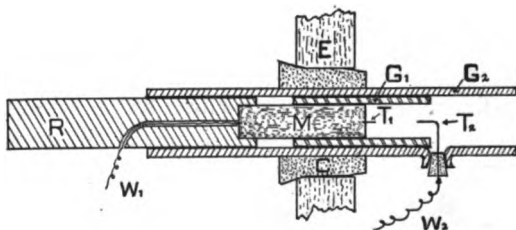


Fig. 2.

the terminals T_1 and T_2 . T_1 is a short piece of No. 24 magnesium wire in the end of a metal cylinder M which is fastened in the end of a hard rubber rod R about 1.4 cm. in diameter and 10 cm. long. A wire W passing through a hole drilled in the rubber rod R connects the metal cylinder M , and therefore T_1 , with one terminal of the electric machine. G_1 is a glass tube .7 cm. in diameter and 5 cm. long, fitting tight in a glass tube G_2 , which is 1.4 cm. in diameter and 18 cm. long. G_2 passes through a cork C in the end E of a wooden box made of inch lumber. The rubber rod R slides back and forth (like a piston) in the tube G_2 , and thus moves M back and forth in the tube G_1 , and so changes the length of the spark gap T_1, T_2 . Fig. 1 shows how R is connected to the lever arrangement L_1, L_2 , which passes to the other end of the box and enables

the observer there to change the light gap at will while the sparks are passing, and thus regulate the time interval, and consequently the size (diameter) of the sound wave.

T_2 is a piece of No. 24 magnesium wire passing through a cork to the wire W_2 connecting with the terminal of the sound gap. The other end of T_2 is bent at right angles as shown in the figure and adjusted so that the spark takes place along the axis of the tube G_1 . The wire T_2 is so small that its shadow is practically negligible. The source is thus an end on spark—itsself practically a point source. Considerable light must be reflected from the end of the cylinder M and the walls of the glass tube G_1 , the tube acting as a sort of gun to direct the light toward the photographic plate. But inasmuch as the tube is only seven millimeters in diameter and the plate is several meters distant, the entire source is relatively a point source.

The light gap is at one end, the photographic plate at the other end, and the sound gap near the center, of a long light-tight wooden box or tube consisting of three sections, the two end sections telescoping into the central section, all painted black inside. This combination permits of a quick and independent variation of the distance between the light and sound gaps, and between the sound gap and the dry plate.

The central section of the tube is four feet long and fourteen inches square inside. The end sections three feet long and twelve inches square inside, all made from planks one inch thick. Two additional sections, not shown in the figure, were sometimes used to get still greater distances between the gaps. Most of the pictures shown in this paper were obtained with the sound gap about one hundred and thirty centimeters from the light gap, and about a meter from the dry plate.

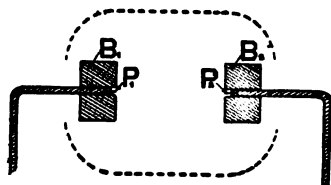


Fig. 3.

The front side of the central section of the box is cut away in Fig. 1, to show the general arrangement of the sound gap.

Fig. 3 shows the details of the gap itself. P_1 and P_2 are the platinum terminals between which the sound spark occurs. These consist of short pieces (.5 cm. long) of No. 20 platinum wire soldered in the ends of brass rods bent as shown. The brass rods are held by wooden blocks arranged to slide snugly between two cleats or guides fastened on a wooden base, permitting an easy adjustment of the length of the sound gap. The two halves of the sound gap are adjusted so as to be in the same plane with the light gap so as to give but one shadow of the terminals on the photographic plate, and also, as will be explained later, to give the best shadow of the sound wave.

The sound gap must be made long enough and the intensity of the spark great enough to produce a strong sound wave, if a distinct wave shadow is to be obtained. Consequently more or less light is produced at the sound gap, where it is not wanted. Moreover the sound gap is much nearer the dry plate than is the light gap. Fogging of the dry plate by light from the sound gap is prevented by three things:

First. The intensity of the light produced between the platinum terminals of the sound gap is far less than between the magnesium terminals of the light gap in series with it. Interchange the platinum and magnesium terminals and the plate is fogged by the light from the sound gap and the light gap fails to give enough light to produce clear shadows on the photographic plate.

Second. When the sound gap is in its usual position, axial with wooden box and light gap—its light is intercepted by one of the two hard rubber cylinders or buttons, B_1 , B_2 , shown in section in Fig. 3. In the axial position the buttons and their supports cast but one shadow on the plate, as seen in all the accompanying sound wave pictures except the first two.

Third. The walls of both the end sections of the box are provided with numerous tin vanes, only one, V , being shown in Fig. 1. These vanes are made of tin strips about an inch wide and twelve inches long, painted black, with one edge tacked to the inside of the box, the other edge being bent inward from the walls of the box so as to point toward the light gap. These vanes intercept almost all the light that otherwise might be reflected by the walls of the box and produce fogging of the plate.

The photographic plate P is carried in an ordinary 8 inch by 10 inch plate holder placed in the reversible back taken from an 8×10 camera. The end of the box was arranged to fit the camera back, light tight, in either position, of the plate. When the plate holder is removed the sound wave shadow falls directly on the ground glass of the camera back, and is distinctly seen by an observer looking through the glass along the axis of the tube; that is, with the eyes in line with the light and sound gaps and the center of the ground glass. If the experiment is carried on in a darkened room the wave shadow on the ground glass can be seen at a distance of several meters even when the glass is viewed obliquely, therefore the waves can be seen by several observers at the same time. Indeed the camera end of the box may be removed entirely and the shadow be allowed to fall on a white screen, or better, on a large sheet of white paper, and the waves be seen by reflection. The author has used this method in exhibiting the waves simultaneously to a number of observers. The screen is placed so the light strikes it obliquely. This

gives a distorted image of the waves, but it enables one to seat all the observers in a single group in front of the apparatus. But little time has been given to the problem of exhibiting sound waves on a large scale to large classes. The partial success attained leads the writer to believe that it can be done. Further details may be expected later.

In Fig. 1 the connecting wires are shown looped in the conventional way in order that the connections may be easily traced. In the experiment the wires were run through glass insulating tubes and as nearly straight as possible. Most of the connections were soldered. In fact many precautions must be taken if one is to obtain even an approach to regularity in the time interval between the sound and light sparks. One must be able to regulate the time interval and maintain it approximately constant, or he may expose photographic plates by the dozen and not once catch the sound wave in the position desired. By experience the writer finds that the following conditions give the best results:

The connecting wires should be of ample size. Small wires (No. 30) appear to slow down the spark and so decrease the intensity of the sound wave and the brightness of the light spark. And the tendency of the discharge to jump from the wires is much greater than when larger (No. 14) wires are used.

The Leyden jars, particularly those on the electric machine and in multiple with it, should not have the usual dangling chain for connection with the inner surface. The connection between the links of the chain and between the chain and the metal lining of the jar is entirely too haphazard to produce the regularity of charge and discharge required in this experiment. The dangling chain is a makeshift in any case.

In this experiment the jars were of the type frequently furnished with large oscillator sets. The short metal rod and dangling chain were replaced by a long rod—reaching almost to the bottom of the jar. On the end of the rod there was soldered a star-shaped disk of thin spring copper with the several radial arms bent so as to make spring contact with the inner tin foil coat of the jar. The results obtained with such jars are far more regular than those which are obtained when using chain connection jars.

From Fig. 2 it will be noted that the outer glass tube G_2 extends some distance beyond the limits of the gap and inner tube. The extension is not primarily for the purpose of reflecting the light toward the sound gap—the gun effect previously mentioned. It is to prevent rapid changes in the gas about the spark gap. Consider what happens when a spark occurs—at the sound gap for instance. The heavy discharge heats the air to a high temperature and convection currents are set up. Besides,

the spark itself violently disturbs the air because of its explosive character. If the spark is in the open the heated and ionized air mixes readily with the air about it so that a second spark takes place in air of varying character. The degree of ionization of the air about the gap depends then upon the time between successive sparks. If the spark takes place in a tube, as shown in Fig. 2, the air about the gap is prevented from mixing rapidly with the air that has not been ionized. Rapid changes in the resistance of the light spark gap are thus prevented and a more nearly constant time interval between the sound and light sparks made possible. At best this interval cannot be made exactly constant. Our knowledge of sparks and spark gaps is too limited. Some of their actions appear to be quite erratic.

There is little need of giving dimensions of capacities, spark gaps, etc., as these may vary greatly among themselves and still give results. Moreover the arrangement that appears to give the best results one day does not necessarily give the best results another day, when the atmospheric and other conditions are entirely different.

Let us suppose the capacities L , L_0 , of Fig. 1 consist respectively of three Leyden jars, including the one on either terminal of the machine. Besides the Leyden jar capacity there is the added capacity of the machine itself, which is considerable in the case of a machine of the size and construction used in this experiment. Suppose further that six Leyden jars are used as the capacity K_1 and that the sound gap S is made about 3.5 cm. long. The speed of the electric machine is then adjusted so that sparks will almost, but not quite, pass when the gaps TK and T_0K_0 are made, say, six and seven cm. long respectively. As explained elsewhere a spark will then occur when the rod R is rotated so as to bring the edge of the glass plates, G , G_0 , between the spark knobs. Sparks continue to pass if the glass plates are not immediately rotated out of the field. By a rapid to and fro rotation of the plates single sparks can be produced at will. This may be done by the observer himself, but preferably by an assistant. The observer stations himself so as to view the ground glass normally and by pulling the lever L_1 makes the light gap very short, say 1 cm. long. The assistant then produces sparks at regular intervals, say 1.5 seconds. If these intervals are not approximately constant the time intervals between the sound and light sparks will vary greatly due to variation in the ionization of the air about the spark gaps, the potential of the Leyden jars, etc. At every spark the observer sees the shadow of the sound gap terminals and, if the interval between the sound and light sparks is within the proper limits, he sees also the shadow of the sound wave produced at the sound gap. If he sees no

sound wave at all, but sees about the shadow of the terminals a cloud resembling a puff of smoke, it indicates that the time interval between the sound and light gaps is too long, that the sound wave has passed beyond the limits of the field and that the hot air from the spark has had time to pass out from behind the sound gap terminals, as shown in Photographs 9 and 10. The time interval must be made less. Since the length of the light gap is already a minimum the interval should be decreased by decreasing the capacity K_1 by removing one or more jars.

If the observer sees neither sound wave nor cloud about the sound gap terminals it indicates that the interval between the sound and light sparks is so short that the sound wave has not had time to emerge from behind the terminals—rather from behind the cylindrical buttons shown in Fig. 3. The observer readily increases the interval by pushing the lever L_1 and thus lengthening the light gap. If the latter is made much longer than the sound gap, sparks will not pass at the light gap. The best illumination is obtained when the light gap is about two thirds as long as the sound gap, say about 2.5 cm. long. If at this length the interval between the two sparks is still too short to show the sound wave, the interval is increased by increasing the capacity K_1 . To get the best results it may be necessary to vary the lengths of the machine spark gaps and possibly the length of the sound gap. The latter is reached through a door or lid in the central section of the box. The effect of changing the length of these gaps is discussed earlier in this paper. Several adjustments and trials may be necessary before the sound wave appears. When it does appear it repays all the time spent in trying to get it, for—if the conditions under which the wave is formed are at all favorable—the wave shadow will stand out with surprising definition and brilliancy whether reflected, refracted, or diffracted.

At first thought it might appear that the sound waves are spherical and that their shadows should be disks, not circles, as shown in the accompanying photographs. In the case of a linear spark the waves are not spherical, but cylindrical—with curved ends. In Fig. 3 the dotted lines represent a longitudinal section of the cylindrical wave that one would expect from a spark between the points P_1, P_2 . The cylindrical buttons B_1, B_2 keep the wave cylindrical until it passes beyond them. The wave then takes on curved ends due to diffraction, but the intensity of the wave at the ends is far less than on the sides. In Fig. 3 the intensity of different parts of the wave is indicated by the breadth of the dotted lines. That this representation is valid is shown by photographs 1 and 2. To obtain these pictures the sound gap was turned at right angles to the position shown in Fig. 1; that is, the sound spark was at right

angles to the axis of the tube and therefore perpendicular to the path of the light from the light gap.

A moment's consideration will show why the wave shadows are always cross sections of the wave—why, in all the pictures except the first two, we do not see any shadow of the ends of the cylindrical wave. In the first place, the energy of the wave is relatively much smaller at the ends than at the sides. This means that the change in the density of the air is less at the ends than at the sides, and that light waves in passing through the ends of the sound wave suffer less change in their velocity than when passing through the sides of the wave.

In the second place, whatever deviation is produced in the light ray in passing through one end of the cylinder is largely counter-acted by the opposite deviation produced in passing through the other end.

The only rays of light that undergo uncompensated prismatic deviation are those that lie just inside the surface produced by drawing a system of tangents from the point light source to the sound wave. These light rays enter the sound wave with a large incident angle, pass along approximately parallel to the sound wave front without passing through the inner surface of the wave at all, and pass out near the other end of the lateral surface at a large angle of emergence. Such rays are deviated more or less and so change the otherwise uniform distribution of light on the plate, giving a shadow of that portion of the sound wave. In any case the only parts of a sound wave that give strong shadows are those to which tangents can be drawn that are parallel (approximately) to the light rays from the light gap. Thus the shadow closely represents a meridian section of the sound wave at right angles to the path of the light from the light gap.

In the early part of this investigation and before the light gap described in this paper was devised, single sparks at the light gap failed to give sufficient light to produce strong negatives. The most sensitive plates of four different manufacturers were tried—with little difference in the results. Indeed the differences between rapid and slow plates exposed to the light of a spark between magnesium terminals is far less than it is when they are exposed in the usual way, to a light much less intense but which lasts hundreds, perhaps thousands, of times as long—even when the exposure is made with a so-called instantaneous camera shutter. Most of the work of this experiment was done with Cramer's Crown plates and a strong hydrochinon (contrast) developer. To save time the plates were developed and fixed in an Eastman tank, a dozen plates at a time. The time of development varied from ten to twenty minutes.

So far as the writer knows all previous work on the visualization or

photography of sound waves has been by the so-called "Schlieren Methode" devised by A. Toepler,¹ and used by Mach,² M. Toepler³ and Wood.⁴ The Toepler Schlieren method is exceedingly ingenious and has very wide application. However, compared with it, the method described in this paper has some obvious advantages.

In the first place no high grade lenses or concave mirrors are required. All optical part are dispensed with.

In the second place the size of the sound wave is not limited by the diameter of a lens or concave mirror, but only by the size of the photographic plate or screen on which the wave shadow is to fall. Since the sound wave can be large, so can the sound lens, the sound gratings, etc.

In the third place, the wave shadow itself may be seen and photographed full size. By the Schlieren method the wave images are small, usually but a few millimeters in diameter. This makes it difficult to study the waves visually and impossible to exhibit them to a class. This defect cannot be overcome by changing the lens system, for the size of the images is limited by the lack of light, most of the light being cut off by the diaphragms which the Schlieren method requires.

In the fourth place the method in this investigation gives a field of uniform illumination and distinct sound wave shadows in all points of the field. The Schlieren method gives a field of varying and uncertain illumination and wave images much more definite at some than at other points of the field.

DETAILS OF SOUND WAVE PHOTOGRAPHS.

Photograph 1. A longitudinal section of a sound wave from a linear spark at right angles to the axis of the tube, showing that the wave is cylindrical with approximately hemispherical ends. (Compare with Fig. 3.) In this position the light from the sound spark falls directly on the photographic plate and fogs it, and on this plate the light from the light gap superposes the shadow of the sound gap and the sound wave. Distinct sound wave shadows cannot be expected under such conditions. However the photograph shows that the wave form corresponds with the theoretical form shown in Fig. 3.

Photograph 1 shows also that *the luminous discharge in air at normal pressure is striated*. On the original negative the striations are quite distinct. Moreover, besides the principal sound wave the negative shows other systems of sound waves. This method of study promises to

¹ A. Toepler, Pogg. Am., 131, p. 33, N. 180, 1867.

² E. Mach, Sitzungsber. d. k. Akad. d. Wissensch. zu Wien, 98, p. 1333, 1889.

³ M. Toepler, Ann. d. Phys., 14, p. 838, 1904. Also Ann. d. Phys., 27, p. 1043, 1908.

⁴ R. W. Wood, Phil. Mag., 48, p. 218, 1899. Also Phil. Mag., 50, p. 148, 1900.

enlarge our knowledge of the conditions attendant upon the spark discharge in air. The writer will soon publish the result of his investigation of the subject.

Photograph 2. This photograph is the same as photograph 1, except that the direct light from the sound spark gap is prevented from falling on the photographic plate by placing between the gap and plate a small rectangular block of wood supported on a metal rod.

Photograph 3. A sound wave just emerging from behind the spark terminal. For this photograph and all the others except Nos. 1 and 2, the spark was axial with the tube, in the position shown in Fig. 1.

Photographs 4, 5, 6, 7, and 8 show the wave as it gradually travels outward from the source until it finally passes beyond the limits of the plate.

Photograph 9. Here the sound wave has passed beyond the limits of the plate and the hot air produced by the sound gap has had time to puff outward from behind the spark buttons. It will be noticed that in photograph 8 the puff has just begun to emerge from behind the buttons.

Photograph 10. The puff at a still later time than shown in Photograph No. 9. The photograph shows how clearly changes of temperature and pressure are recorded by this method.

Photographs 11 and 12 show a wave reflected at the plane surface of a piece of glass plate about 10 cm. wide and 20 cm. long, held vertically a short distance from the spark gap, with the surface of the plate parallel with the gap.

Photograph 13. This shows a wave (cylindrical) reflected from a concave cylindrical reflector. The reflector was made from a piece of lead about 6 cm. square bent around a gas pipe and held at the end of a vertical strip of sheet brass to which the reflector was soldered. Note the reflected wave front, and the reflected and diffracted waves about the edges of the reflector which are parallel to the spark gap.

Photograph 14. Wave reflected from a convex surface, here the surface of a cylindrical convex lens as described in the next paragraph.

Photograph 15. A plane wave produced by placing the spark gap at the principal focus of a convex lens of sulphur dioxide gas. The lens was constructed as follows:

A block of wood, *W*, Fig. 4, was shaped like a cylindrical lens about 10 cm. long, 4 cm. wide, and of the thickness desired,—almost 2 cm. in the case of the lens used in photograph 15, more than 3 cm. in photographs 16, 17 and 18. To form the lens space *L* a section 5 cm. long was cut from the center of the block, except for a narrow strip *S* at the bottom, to hold the end sections in place. A rubber band *B* was stretched between

the two extreme ends at the upper edge of the block and a portion of the thin edge of the block cut away just under the band so the band might be as free to move as possible. The sides of the block near the ends and the lower cross strip (the surface shaded in the figure) were coated with a light coat of thin shellac varnish. Before the shellac had time to dry the block was covered with a single collodion film with the lap on the wood strip *S*, and everywhere made gas tight by the shellac; thus forming between the collodion surfaces a cylindrical gas lens, *L*. The rubber band *B* was to keep the surfaces in position and still permit of considerable distortion of the walls of the lens without tearing the film. This was not

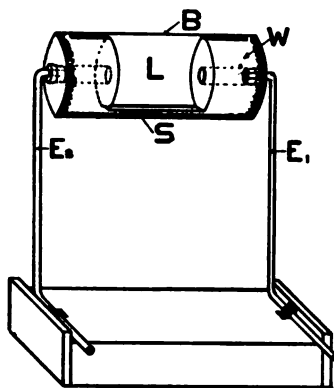


Fig. 4.

possible when the films were pasted on lens forms having both an upper and lower rigid strip (*S*) and when the films were pasted to the form up to or close to the lens space *L*. A very few sound sparks invariably sufficed to rupture the films on such lenses, the rupture always being on the side away from the sound spark. There was no rupture when using a lens of the construction shown in Fig. 4. The points of attachment of the edges of the film are so far apart that the film has considerable freedom of motion, and is therefore not so easily ruptured.

Obviously the freedom of motion is still further increased by the use of the rubber band *B*.

Collodion films are exceedingly strong considering their thickness, but they are rather stiff and they will not stretch without tearing. The writer found that a film made of so-called "liquid skin" is more flexible and extensible. The formula for the liquid skin is: 92 gm. collodion, 5 gm. Canada balsam, 3 gm. castor oil. When thoroughly mixed and free from air bubbles a small quantity of the fluid is mixed with three or four times its volume of ether. This is then poured into a glass dish with a flat bottom, or on a piece of glass plate, and the dish or plate drained at once by standing on edge. When the film is thoroughly dry it is floated off in the usual way—by allowing water to creep in between the film and the glass. The film should dry thoroughly before one attempts to mount it.

The gas is introduced into the lens *L* through a small glass tube *E*₁ connected with the gas generator. The gas escapes from the lens by means of the glass tube *E*₂, to which a long rubber tube is attached so that the gas can be passed into a hood or out at a window. The tube *E*₂ is

larger than E_1 , so that there may be no danger of the gas pressure increasing in L , and rupturing the film. A continuous flow of gas through the lens was maintained whenever the lens was in use. The sulphur dioxide gas was generated by the action of hot concentrated sulphuric acid on copper and the flow of gas controlled by regulating the heat. The pressure of the gas in the generator was maintained at about 5 cm. of water, as indicated by a manometer attached to the generator.

All of the lens photographs show the original wave, the wave reflected from the convex surface of the lens, the refracted wave, and the wave diffracted about the upper and lower edges of the lens. The film of the lens in photograph 15 has a minute perforation in it through which the SO_2 gas escapes. The path of the escaping gas is clearly indicated.

Photograph 16. A sulphur dioxide lens—thicker than the lens of photograph 15. Here the divergent sound wave has been rendered convergent.

Photograph 17. A carbon dioxide lens. Here the refracted wave is still divergent; that is, the spark gap is nearer the lens than the principal focus.

Photograph 18. A hydrogen lens. Here, contrary to what one should expect from the velocity equation, the refracted wave has suffered practically no change in curvature. In fact the several pictures appear to indicate that the velocity of the sound wave of an explosive character is not given by the Newton equation. This fact has been observed by others.¹ The author will discuss the subject more fully in a later paper.

Photographs 19 and 20. Sound waves reflected from a parabolic reflector made from a sheet of lead 6 cm. wide and 20 cm. long and supported by an upright strip of brass as shown in the photographs. The spark gap was placed at the focus of the parabola. The reflected wave is plane except (photograph 20) for the disturbance produced by the hot air through which the central part of the wave must pass.

Photographs 21, 22, 23 and 24 show four different positions of the sound wave reflected by an elliptical reflector. The reflector was made from a strip of sheet lead about 5 cm. wide and 25 cm. long bent into the form of an ellipse and the two ends of the lead sheet soldered together. It was supported on a block of wood cupped to fit it. The brass rods that support the spark terminals are so far apart (see Fig. 3) that the lead reflector is readily placed in position between them. The sound wave is produced at one focus and in photograph 24 it has almost reached the other focus. The expected symmetry of the direct and reflected waves at the center of the ellipse is shown nicely in photograph 22.

¹ P. Vieille, *Comptes Rendus*, 127, p. 41-43, 1898.

Photographs 25, 26, 27 and 28 show four positions of a sound wave reflected from and transmitted by a diffraction grating. The grating was made by cutting four equal and equally spaced rectangular slits (O, Fig. 5) in a strip of sheet tin 6 cm. wide and 18 cm. long. The slits were 7 mm. wide and 3.5 cm. long, with a strip of tin 7 mm. wide between the openings. The tin was tacked to a wooden block which served as a supporting base. The grating is placed with its apertures parallel with the spark gap. In this position the shadow on the photographic plate is edge on and the location of the apertures is not shown on the plate. To correct this fault pieces of heavy wire (W, Fig. 5) were soldered to the upper, lower and intermediate strips of tin, so that an edge on shadow of an aperture is narrow, of a reflecting surface it is broad.

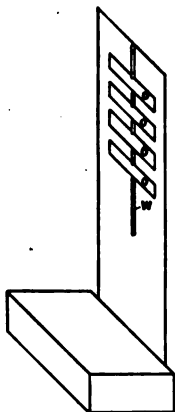


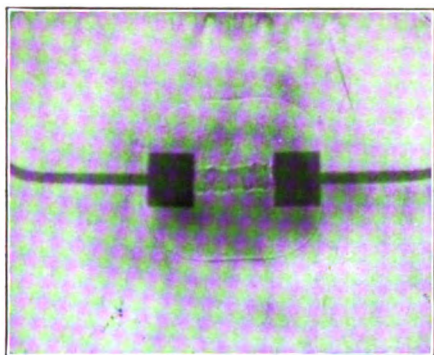
Fig. 5.

Both the reflected and transmitted system of waves are in complete accord with Huygens's principle. It seems to the writer that such photographs as these, or better, a chance to see the waves themselves, might be used by a teacher to give a concrete and definite idea of Huygens's principle and deductions from it, to students who have difficulty in forming mental images. Reality would be given to such terms as secondary waves, center of disturbance, pole of wave, wave front, common tangent, diffraction, etc.

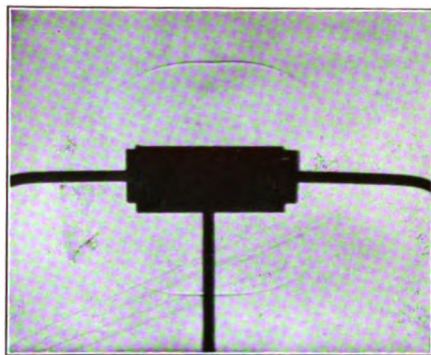
Photographs 29 and 30. These pictures were made by substituting a curved grating for the plane grating of the last four photographs. The grating is similar to the plane grating already described except that it has eight slits instead of four, and is cylindrical—having been curved over a piece of gas pipe. It was intended that the spark gap should be on the axis of the cylinder but it was accidentally displaced slightly before the photographs were taken. This explains why the transmitted and reflected wave systems are not symmetrical with respect to the grating and the spark gap.

All the photographs having a reflected wave show that a condensation is reflected as a condensation. This is true in the case of the hydrogen lens. While the film separating the rare hydrogen and the denser air is exceedingly thin, nevertheless it is a dense body so far as sound wave reflection is concerned. In a later paper the author will show waves reflected from the boundary between two gases of different density—when there is no film or other object separating them.

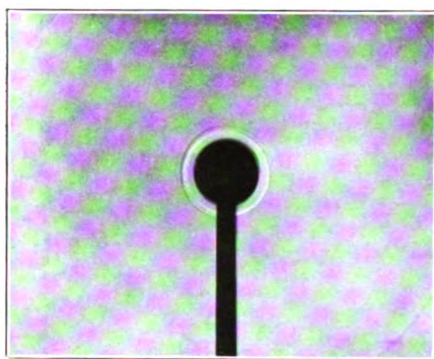
PHYSICS LABORATORY,
INDIANA UNIVERSITY,
June, 1912.



Photograph 1.



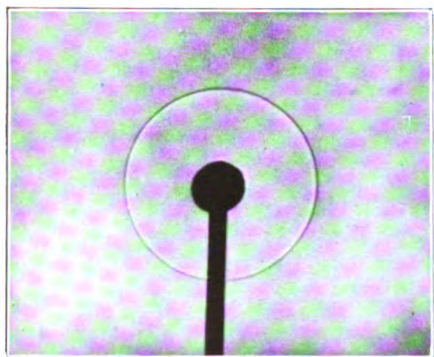
Photograph 2.



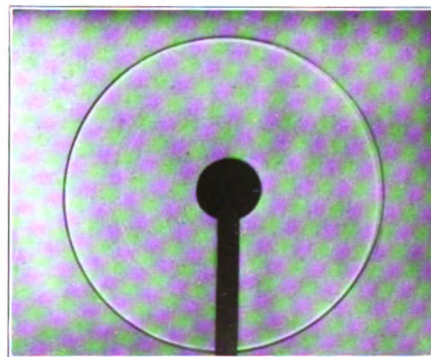
Photograph 3.



Photograph 4.

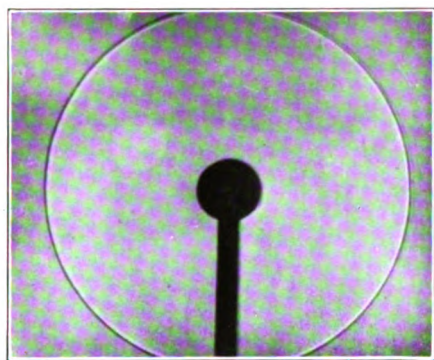


Photograph 5.

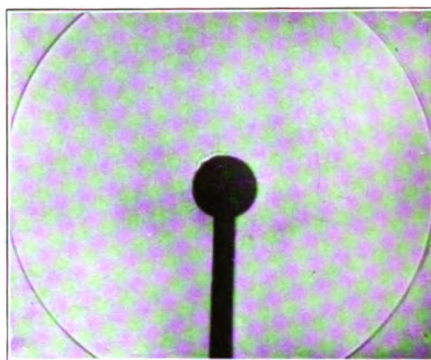


Photograph 6.

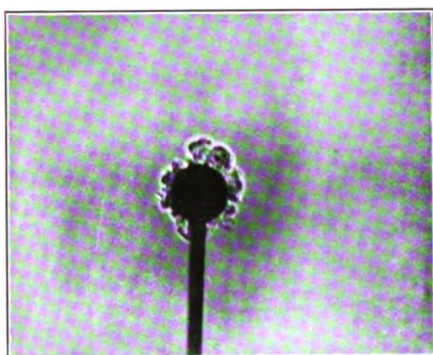
A. L. FOLEY AND W. H. SOUDER.



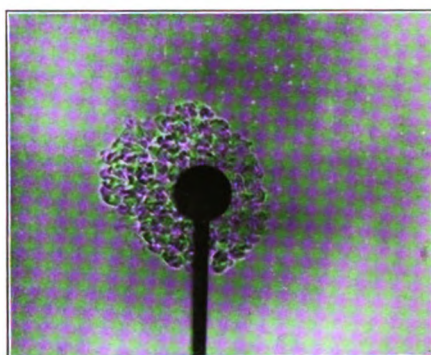
Photograph 7.



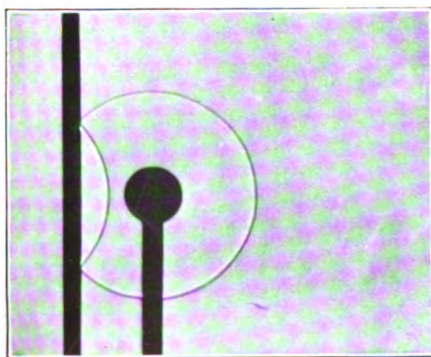
Photograph 8.



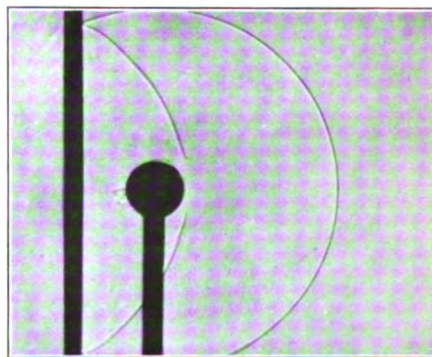
Photograph 9.



Photograph 10.

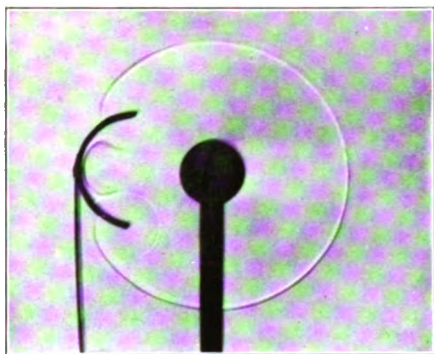


Photograph 11.

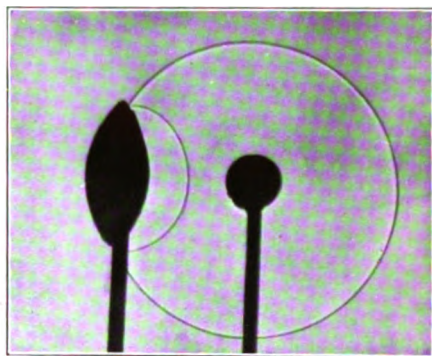


Photograph 12.

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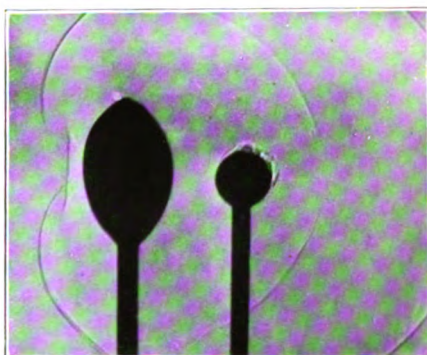
Photograph 13.



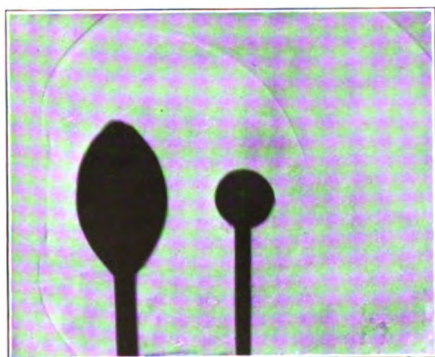
Photograph 14.



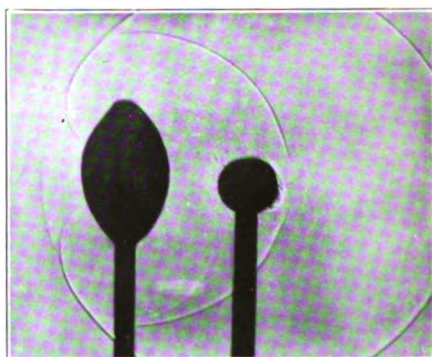
Photograph 15.



Photograph 16.

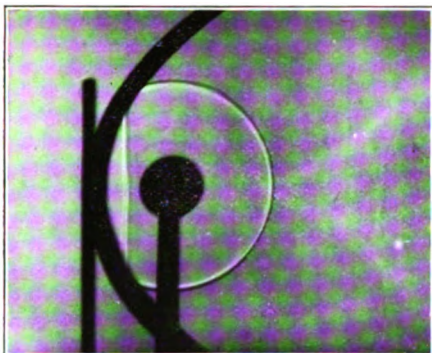


Photograph 17

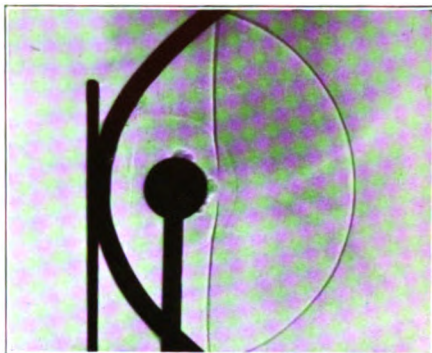


Photograph 18.

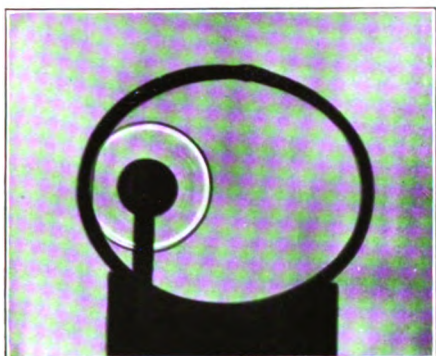
A. L. FOLEY AND W. H. SOUDER



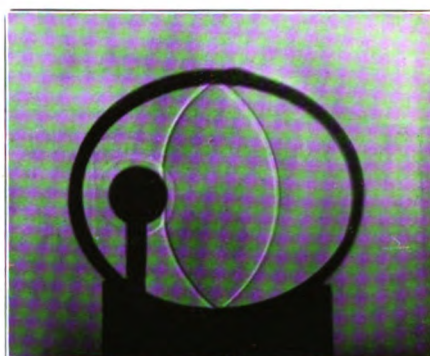
Photograph 19.



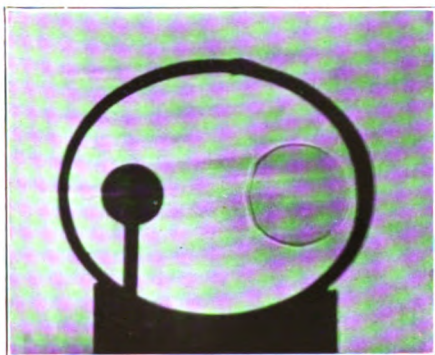
Photograph 20.



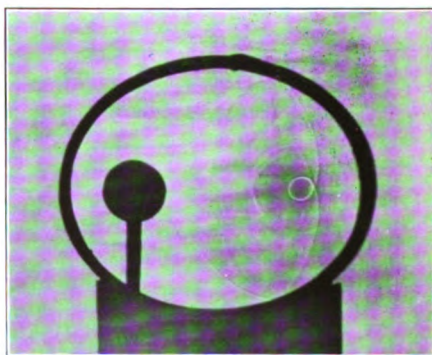
Photograph 21.



Photograph 22.

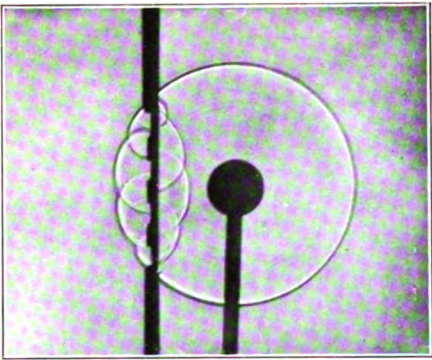


Photograph 23.

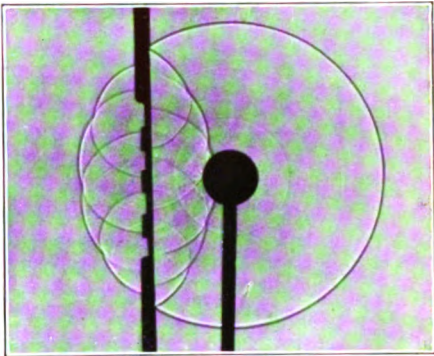


Photograph 24.

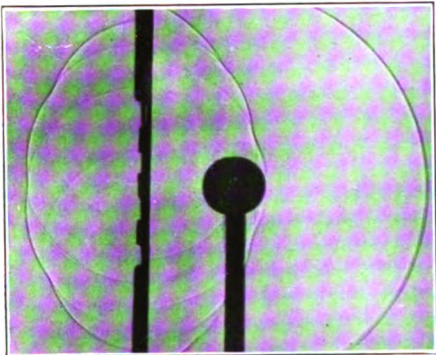
A. L. FOLEY AND W. H. SOUDER.



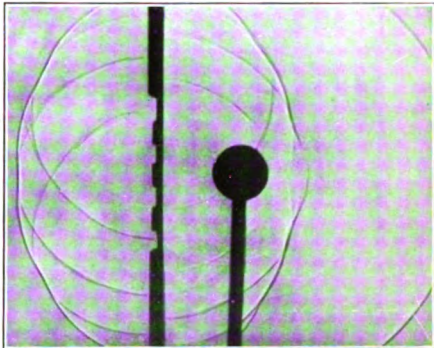
Photograph 25.



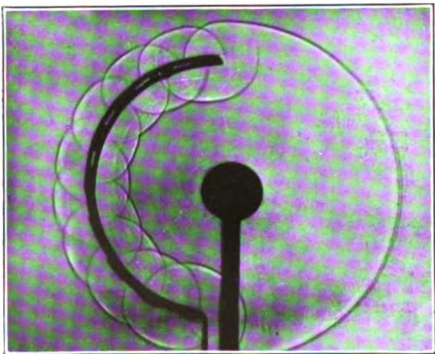
Photograph 26.



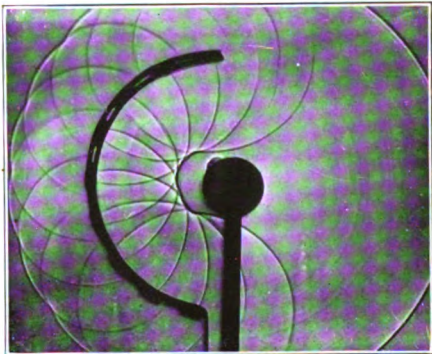
Photograph 27.



Photograph 28.



Photograph 29.



Photograph 30.

A. L. FOLEY AND W. H. SOUDER

OSCILLOGRAPHIC STUDY OF THE SINGING ARC.

By J. E. HOYT.

THE line of work taken up by the author in this paper was suggested by the remark of the editor of the *London Electrician* (Vol. 46, p. 356) in reference to the Duddell arc. "One feature of this experiment was brought out in this paper, viz., there is an alternate current generated by the arc of the same periodicity as the note emitted, but the experiment failed to make this clear. We would like to know if a telephone receiver placed in the circuit of the arc would emit a note that might be analyzed into only the same elements as the tone set up in the arc itself, or whether there are current variations set up other than those to which the shunt causes the arc to resound. In other words if Mr. Duddell's musical instrument were introduced into a telephone circuit, would the receivers in the circuit play the same tune as the arc itself, or would a jangle of other sounds be superposed upon the tune? Another question, if two or more arcs in series were to be separately shunted for different tones, would a telephone in series with all the arcs emit the chord combining the several tones?" No reports of work done along that line have come under this author's observation. Therefore it occurred to him that a qualitative study of the wave forms and of the harmonics present in the oscillations, and their relationships to the various conditions of the shunt circuit, of the arc itself and of the inductance in the source circuit, would be of interest. It was planned to make an oscillographic study of the resultant effect of two or more parallel circuits shunted about the arc, each with its own capacity and inductance, recording the current in the shunt circuits and in the arc at the same time.

The arc that was used in these experiments was an ordinary hand feed, vertical arc, capable of various adjustments, one of the electrodes being water cooled. The water furnished an atmosphere of vapor and produced stronger and more constant oscillations. Unless otherwise specified the electrodes were solid carbons 6-8 mm. in diameter. In the source circuit was usually connected as chokecoil, the primary of a transformer, and also a bank of incandescent lamps in parallel. The oscillograph was one of the G. E. type, having three bi-filar suspension systems. These were calibrated so that volts or amperes could be read directly from the curves. For the inductance of the shunt circuit there were two coils of

300 turns each of No. 10 wire, wound on cylinders 30.3 cm. in diameter and 93 cm. in length. The total resistance of one of these coils was about .9 ohm. The inductance was computed from Russell's formula,

$$L = (\pi dn)^2 l \left[1 - 0.424 \left(\frac{d}{l} \right) + 0.125 \left(\frac{d}{l} \right)^2 - 0.0156 \left(\frac{d}{l} \right)^4 \right],$$

and was found to be $6.82 (10)^6$ cm. Connection could be made at various points so that the total induction of the coil could be varied by steps of $2.27 (10)^6$ cm., and a bare wire coil could be connected in for finer adjustment. Three standard boxes of 10 mf. capacity each were used. Thus the lowest frequency calculated from Duddell's formula (which gives approximate values for low frequencies) would be 330 oscillations per second. For oscillographic study of course a low frequency was necessary. A preliminary investigation of the reliability of the stroboscopic and acoustic methods of determining the frequency of the electric oscillations was first decided upon. For the former a Duddell falling plate oscillograph was called into service using the singing arc itself as source of light. After overcoming very many difficulties a record was obtained which unfortunately is too faint for satisfactory reproduction. However, as might be expected, the spots of light evidently appear most distinct just after the minimum point of the potential difference curve, which corresponds to the instant after the maximum of the current. It is very evident, as was expected, that the variation in light intensity and current are isochronous.

In order to compare the electric and sound vibrations, it was decided to construct a sound oscillograph. This apparatus, as finally set up, consists of the following: Mounted on a heavy horizontal frame, attached to a heavy stand, is a heavy brass elbow, holding a brass frame about four inches in diameter, which in turn holds a fine specially prepared parchment diaphragm. The latter frame is so constructed that the tension of the diaphragm can be increased to the point of rupture. A high tension was used in this experiment. Also rigidly attached to the horizontal frame is a block, bearing on its face a flat section of a thick-walled brass cylinder. In this cylinder vertically opposite each other are two screws, in the ends of which jewel bearings are set, and between them is mounted the needle bearing the mirror. For this needle, the smallest that could be found anywhere was taken, cut in two and the blunt end sharpened. The mirror is about $.03'' \times .1''$. The horizontal "suspension" is of unspun silk attached to the diaphragm at one end, passing around the needle, and attached at the other end to a very light spring, piece of spring brass, or even a very light rubber fiber. In order

to rotate the needle for adjustment of the mirror without changing the tension, the brass frame holding the needle was mounted on a brass platform. Two strips of spring steel held the frame against the block. Then by means of a screw at one side of the frame and a piece of spring steel at the other, the frame could be moved back and forth along the platform thus causing the fiber to rotate the needle without changing the tension. Two screws from the back of the block made it possible to tilt the frame, needle and mirror backward or forward as needed. Since the heat of the arc was much too great to bring near the diaphragm, it was necessary to make use of a parabolic reflector and parabolic megaphone to obtain sufficient energy at the diaphragm. However the acoustic apparatus responds to almost any noise in the room, and vibrates freely in response to the human voice or to a mounted tuning fork across the room.

In order to obtain simultaneous records a G. E. oscillograph was installed and a pencil of light from the third prism was intercepted with a right prism directing it through a hole in the side of the box, upon the acoustic system outside. From this mirror it was reflected back and onto the photographic film. The whole apparatus was connected up as in Fig. 1. The diagram of apparatus shows: *A.S.*, the arc source; *R.P.*, right prisms; *E.S.*, the electric oscillograph system; *A.O.S.*, the acoustic system; *P.M.*, the parabolic megaphone; *S.A.*, the singing arc; *P.R.*, the parabolic reflector. The connections for the singing arc show *S.A.*, the singing arc; *C.C.*, the choke coil; *S.R.*, series resistance in source circuit; *O*, the oscillograph loop; L_1 , C_1 and O_1 represent the inductance, capacity and oscillograph loop for shunt number one; L_2 , C_2 and O_2 have corresponding significance for shunt number two.

It was exceedingly difficult to obtain good records of the sound of the singing arc, for even while the arc is apparently constant as far as the ear could detect, yet the sound might be varying slightly in frequency. Even if the fundamental was constant, the harmonies due to conditions in the arc were changing in relative intensities and thus changing the character of the wave form. Still some very interesting records were

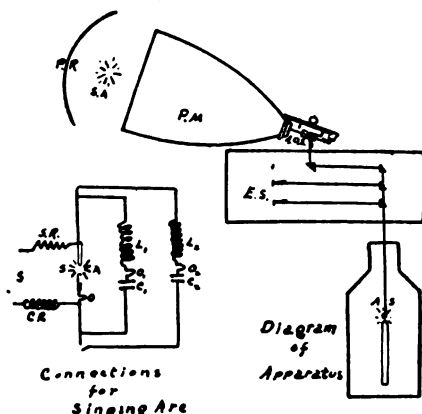


Fig. 1.

obtained, probably the first acoustic records of the singing arc, surely the first obtained simultaneously with the electric records.

Several preliminary records were made of the harmonic vibrations of several tuning forks, vibrating simultaneously. These records suggested a use to which the acoustic system may be devoted,—the obtaining of accurate time values for electric oscillations of low frequency. For example simultaneous records were made of the variation of the arc potential and the sound vibration of Sol₃ fork. Comparing the two it was found that the frequency of the arc was about 420 os./sec. Computation by the simple formula

$$n = \frac{1}{2\pi} \sqrt{\frac{1}{Lc}}$$

gave a value of approximately 430 os./sec. From this it would appear that the simple formula will do for rough approximations in the case of low frequencies, especially where the resistance is very small compared with the other quantities involved.

Fig. 2 is a record of the acoustic oscillations of the arc, the current through the arc and the current in the oscillating shunt. It is very evident that the fundamental of the sound has the frequency of both the arc and the shunt circuit. The latter is reversed in direction, but can be seen to be very nearly the complement of the variation in the arc current, although the current through the arc appears to be more nearly a sine function. A single harmonic appears in the sound vibration, which does not appear in the current curve. This is probably due to mechanical effects of the impact of discharge rather than the electrical conditions. The capacity and inductance of the shunt circuit were 10 mf. and 6.82 (10)⁶ cm. respectively, giving an approximate frequency of 600. The maximum value of the current is a little over 3 amperes.

Some approximate measurements were taken of curves representing the variation of current through the arc and also the variation of sound.

Time in Sec.	Arc Current in Amp.	Relative Values from Sound Curve.
0.	-2.6	-32.
1.6(10) ⁻⁴	+3.0	-15.
6.4(10) ⁻⁴	+0.2	33.
9.6(10) ⁻⁴	-2.0	+ 5.
14 (10) ⁻⁴	-3.1	-10.
15 (10) ⁻⁴	-3.0	0.
18 (10) ⁻⁴	-2.2	-10.
22 (10) ⁻⁴	+2.2	-31.

Fig. 3 represents the arc potential, current in shunt and acoustic vibrations for two shunt circuits, natural frequencies, 210 and 420. The value of the inductance in the smaller shunt is half that in the larger, as well as the value of the capacity. The maximum value of the current which is here the combined current of the two shunts is about 5 amperes, the variation in potential difference about 72 volts. The harmonic in the latter is very pronounced here though the sound was that of the lower frequency, so far as the ear could distinguish; the record, however, evidently shows at least one harmonic present. The fundamental evidently is in phase with the current through the arc, opposite to that of the shunt, but, though the current through this shunt is far from a simple wave form it is impossible to ascribe the harmonics of the sound to variations in the current. Here the current was oscillating quite freely in both shunts but when the arc is responding to the higher frequency the lower frequency shunt is unaffected and the current in the combined shunt—the arc—approximates more nearly the sine form. Figs. 4 and 5 show curves for same quantities under different conditions. In cases where

the natural frequencies of the shunts were related as 2-1, the arc was liable to see-saw up and down, now giving the higher tone, now the lower, but in general the higher tone was more stable at smaller distances between the carbons, while the lower tone was more stable with greater separations. Especially was this true where the relationship was not a simple one. The higher tone was obtained most clearly and was most stable when the separation of the arcs was only .04 inch, but the lower tone was most stable with a separation of .1 inch or more. This would seem to indicate a sort of tuning or resonance effect. It will be noticed in regard to

the sound oscillations that the smoother and more sinuous the current curve, discharging through the arc, the simpler the sound oscillation. This might be explained on the idea of difference of impact in the arc.

Curves were also taken, showing the pd., current, and sound of arc, with no large inductance in the source circuit,—that is, with the choke coil removed. Two successive sets of curves were taken under the same

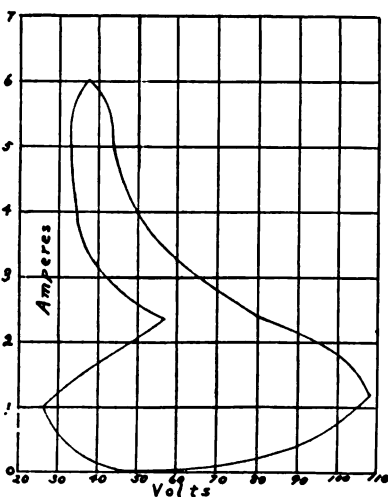


Fig. 6.

Dynamic characteristics. Pd. vs. current through arc.

conditions, only in the second, the coil was again inserted but the two sets are almost identically alike in form. Blondel has noted a difference in the quality of the tone under these differing conditions, but as far as the acoustic record goes there is no considerable difference in wave form though there may have been some difference in period. Each was recorded with the first harmonic, only.

Fig. 6 is a curve showing the dynamic characteristics of the singing arc. This was obtained by projecting simultaneous curves of pd. and current and tracing on a screen. Simultaneous values were then obtained and

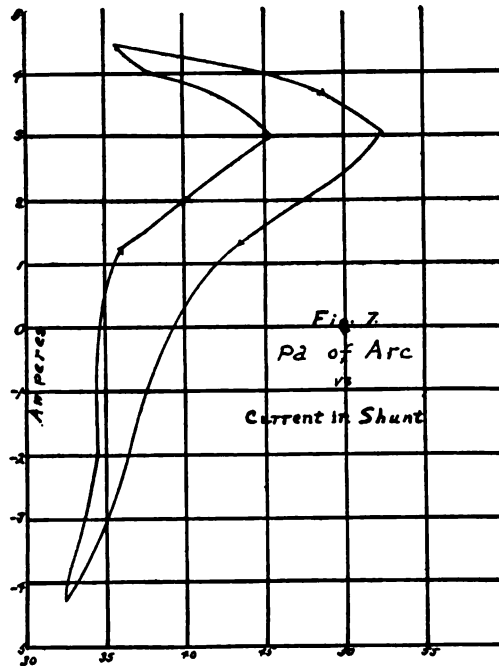


Fig. 7.
Volts.

plotted. It can be seen that the current of the arc actually reaches 0 in this case. Then with slowly rising current the potential rises rapidly, reaching a maximum, then the current rises more rapidly as the potential falls. For a brief decrease of potential, the current falls rapidly, then the potential starts to increase to a second maximum, decrease, then increase until the current reaches 0, when the cycle repeats itself. Thus for the greater part of the curve the dv/di is negative.

Fig. 7 represents the change of potential with variation of current in oscillating shunt. This, however, was with different constants in the

oscillating circuit, about $5.12 (10)^6$ cm. inductance and 30 mf. capacity. It will be noticed that with increasing current and decreasing inverse discharge we have a slight increase of potential for a time, then a more rapid increase, then a rapid decrease of potential with increasing current. With decreasing current the potential increases to a second maximum, then decreases rapidly to a minimum value with maximum discharge of condensers through the arc.

Fig. 8 shows the currents through the arc, and through each of two parallel oscillating circuits. The circuits were arranged to give relative frequencies of $2/1$, when the arc responded to them separately, but they

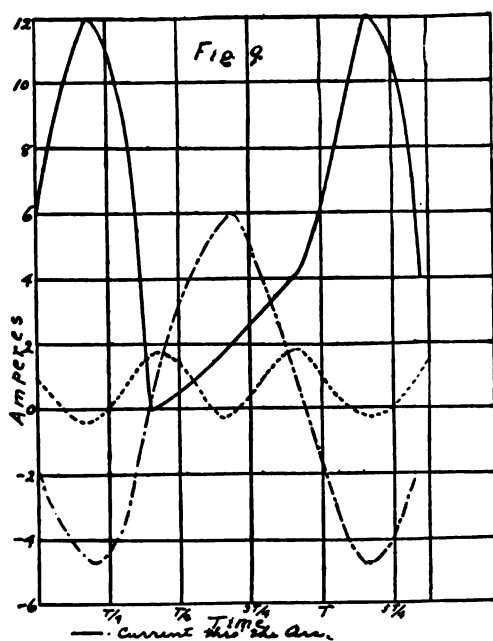


Fig. 9.

Solid line shows current through the arc; dotted line shows current through oscillatory shunt No. 1; broken line shows current through shunt No. 2.

simply produce a resultant effect regardless of the relative frequencies, generally oscillating with frequencies an octave apart. The discharge through the arc is far from a sine function, and the discharge through the oscillating circuit appears much more nearly a sine wave. Fig. 9 shows the relative values of these quantities; i. e., currents in arc and shunt circuits, on a larger scale, drawn from a projection of a negative, on a screen. Apparently from this, the currents in the oscillating shunts are not quite sine functions, though more nearly so than the current through the arc. Apparently the current in the arc should equal the

sum of a constant d.c. and the sum of the discharges due to the shunt circuits. The current through the arc when not oscillating was about 7 amperes. The maximum value of the current in the arc is very nearly equal to the sum of the inverse discharges and this value. But in the neighborhood of its minimum value this does not hold true. Evidently the wave form would be affected by temperature of the carbons, ionization and inductance of the arc. The current is seen to rise very gradually at first then more suddenly as might be expected.

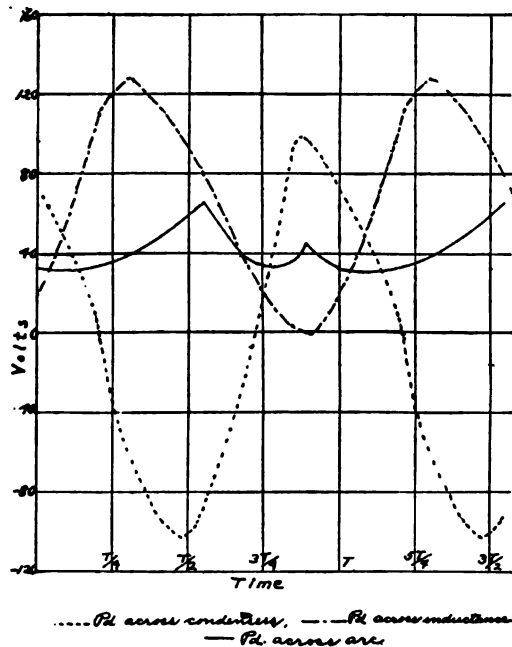


Fig. 11.

It was desired to find the relationship between the arc potential and the variation of potential differences at the terminals of the condenser and the coil in the single oscillating shunt. This is shown in Fig. 10. Fig. 11 shows a similar curve to a larger scale. The great variation in the pds. of the parts of the shunt contrast with that of the arc, the pd. variations of which bear little resemblance in amount or in phase with the individual variations or the vector sum of the pds. of condenser and inductance. It is quite evident that there must be something in the arc itself which influences the variation of potential, more than the conditions of the oscillating circuit. In general when the resulting potential of the shunt was rising the arc potential was falling and vice versa.

Fig. 12 shows the variations of potential across the arc, and across the condensers of the two oscillating circuits when the shunt circuits are tuned to the octave. Fig. 13 shows the same conditions as are represented

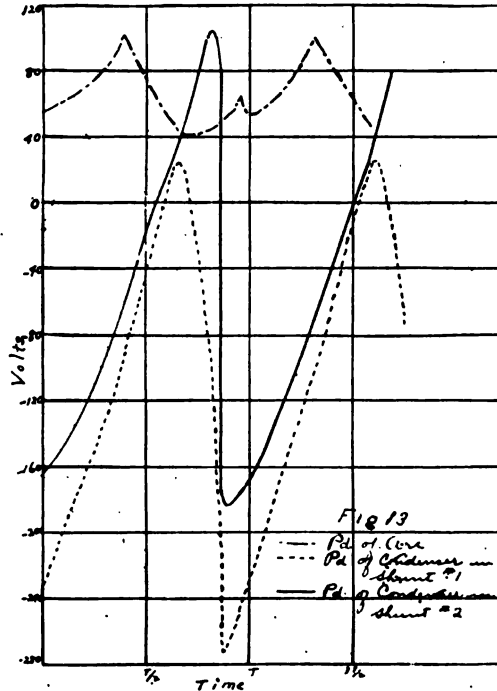


Fig. 13.

in Fig. 12, on a larger scale. The effect of adding another shunt was to produce a further relative phase shift. Furthermore the zero of these shunt curves is not the potential of the negative carbon, but about -45 volts and -120 volts respectively. It will be noticed through what a wide range the oscillations pass. The time of maximum current was not the same for each circuit and the rate of discharge was different, but the time for complete reverse discharge was about the same.

SUMMARY.

Oscillograph curves are here reproduced from photographs, of varied electrical conditions of the singing arc. In the case of a single shunt the current in the arc is very nearly a sine curve and its frequency can be approximately determined from the simpler Duddell formula. The current through the arc is approximately that of the oscillating circuit superimposed upon the d.c. The effect of adding a second shunt is to cause the arc current to deviate largely from the sine form, nor does it

vary as the sum of the shunt inverse discharges. The current is evidently considerably affected by varying conditions in the arc. The arc potential does not even approximately follow the variation of the sum of the pds. of the condenser and inductance. The effect of introducing another shunt is to produce a phase shift in the condenser pds. which may have the same frequency of discharge even if of different natural frequencies.

An acoustic oscillograph was also constructed and herewith are reproduced the first acoustic curves taken of the singing arc, and possibly the first reproductions of simultaneous electrical and acoustic oscillations. Ordinarily for a single shunt it is found that the sound, which is isochronous with the electrical oscillations, is of a rather simple form, the oscillograms showing only a single harmonic. In the case where the arc current is more complex, as when two shunts are used, the resulting sound curve is quite complex. In the latter case the arc responds to one or the other of the shunt circuit frequencies, depending, *e. g.*, on the separation of the carbons. There seems therefore to be a sort of tuning effect, regulated by some electrical changes produced by opening or closing more or less the arc gap. This might possibly be explained by attributing the effect to the presence of induction in the arc, increasing with the separation. This inductive effect might also furnish the basis for an electrical explanation of the change of frequency with arc length which would be quite marked for higher frequencies where the inductance of the shunt is quite small. It is generally accepted that the singing arc discharge is rotational and that the pressure conditions in the arc are very peculiar.

No fundamental difference is recorded in the character of the sound oscillations for the two kinds of tones attributed to the oscillating discharge with and without inductance in the source circuit.

Considerable difficulty was early experienced in making the parts of the apparatus work together. Mechanical vibrations due either to the falling plate or the interaction of the motor and the oscillograph box to which the drum was attached, were hard to eliminate. The author is now designing an improved combined apparatus for the simultaneous recording of sound and electrical oscillations which he hopes to use for telephonic work, testing of diaphragms, etc.

In concluding, the thanks of the author are extended for the courtesy and suggestions of Professors Goodspeed and Richards, also for skilled assistance of Messrs. Jacob Martin and Kalmbach, in the construction of some of the apparatus.

RANDAL MORGAN LABORATORY,
UNIVERSITY OF PENNSYLVANIA,
June, 1911.

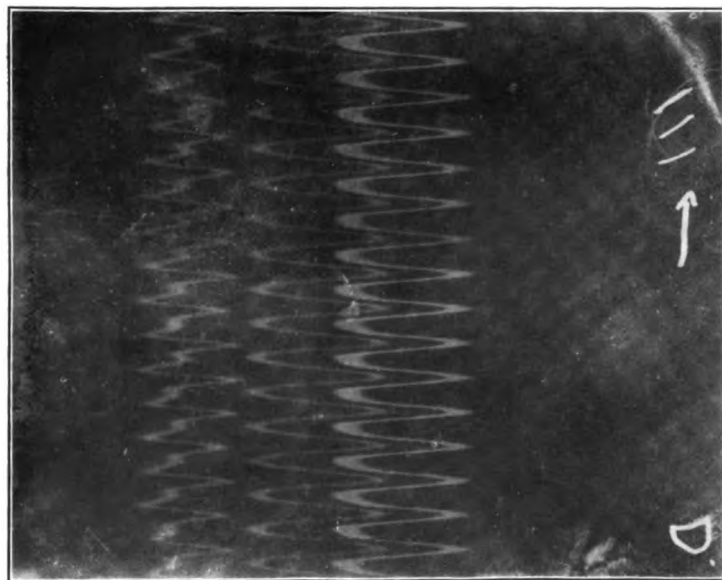


Fig. 2.

I_1 = current in shunt circuit; I_2 = current in arc circuit; A = mf.; L = 682 (10)⁶ cm. acoustic oscillation; C = 10

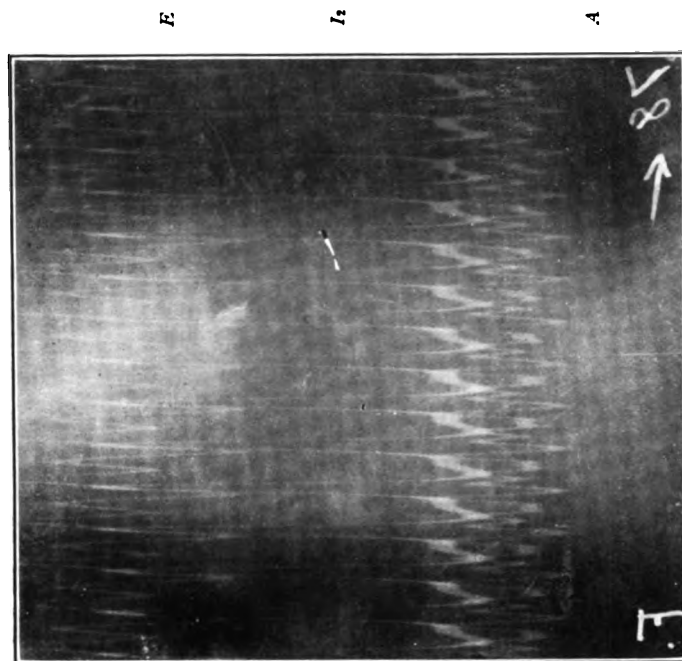


Fig. 3.

E = pd. of arc; I_2 = shunt current; A = sound vibration; C_1 = 10 mf.; L_1 = 341 (10)⁶ cm.; C_2 = 20 mf.; L_2 = 682(10)⁶ cm.

J. E. HOYT

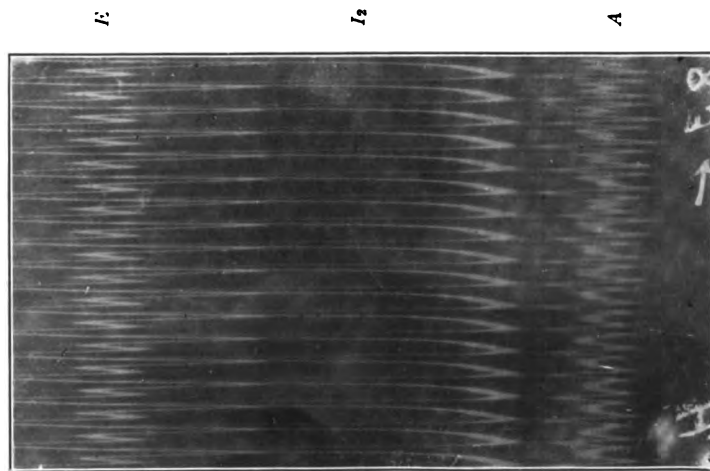


Fig. 4.

$L_1 = 682 (10)^4$ cm.; $C_1 = 10$ mf.; $L_2 = 170 + (10)^4$ cm.; $C_2 = 10$ mf.

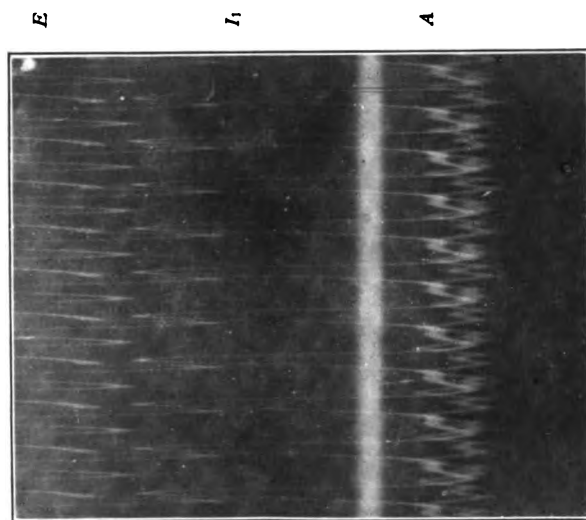


Fig. 5.

$C_1 = 20$ mf.; $L_1 = 682 (10)^4$ cm.; $C_2 = 5$ mf.; $L_2 = 682 (10)^4$ cm.



Fig. 5A.

I_1 = current in arc; I_2 = current in shunt; A = acoustic oscillation;
 $C_1 = 14$ mf.; $L_1 = 682 (10)^4$ cm.; $C_2 = 14$ mf.; $L_2 = 170 + (10)^4$ cm.

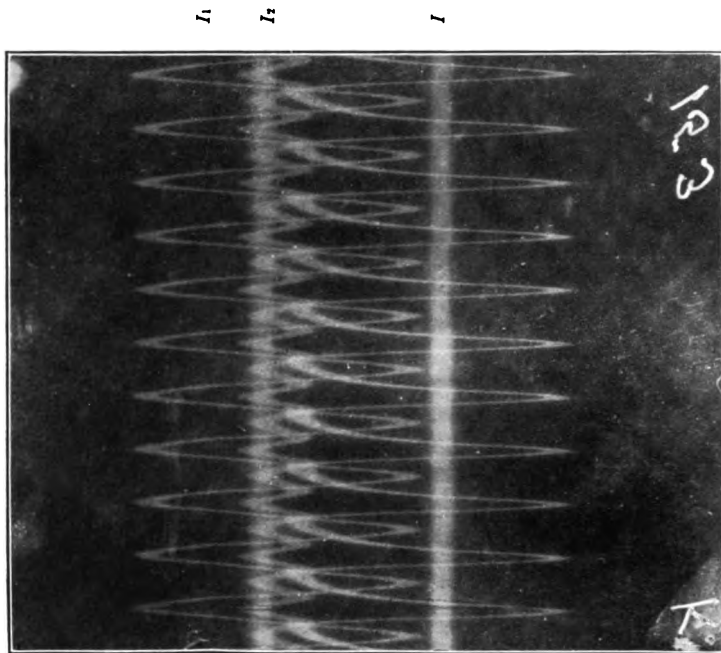


Fig. 8.

I = arc current; I_1 and I_2 = shunt currents; $C_1 = 30$ mf.; $L_1 = 682 (10)^4$ cm.;
 $C_2 = 10$ mf.; $L_2 = 341 (10)^4$ cm.

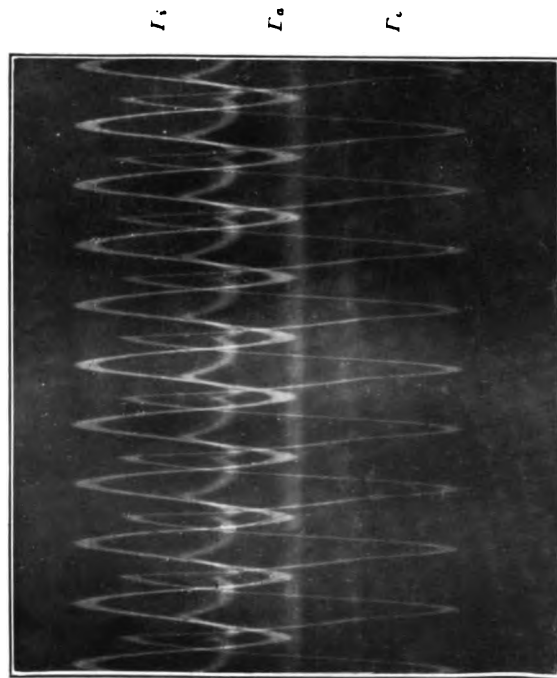


Fig. 10.

E_a = pd. of arc; E_e = pd. condenser; E_i = pd. of coil; C = 20 mf.; L = 682 (10)⁴ cm.

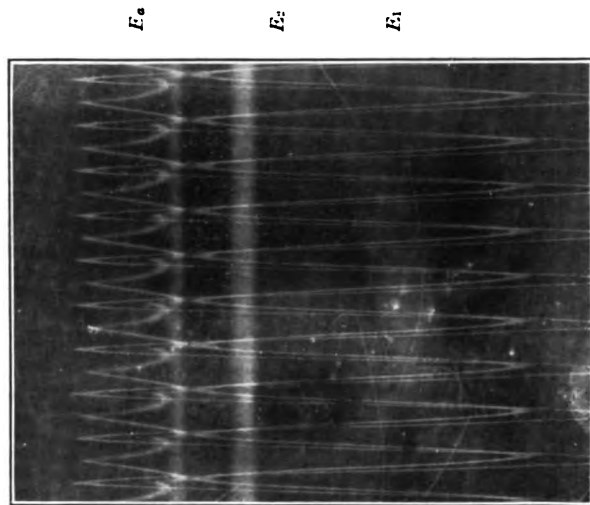


Fig. 12.

E_1 , E_2 = pd. shunt condensers; E_a = pd. of arc; C_1 = 8 mf.; L_1 = 682 (10)⁴ cm.; C_2 = 18 mf.; L_2 = 682 (10)⁴ cm.

PROCEEDINGS
OF THE
AMERICAN PHYSICAL SOCIETY.

MINUTES OF THE SIXTY-THIRD MEETING.

A MEETING of the Physical Society was held in the Fayerweather Physical Laboratory of Columbia University, New York City, on Saturday, October 12, 1912. In the absence of the president and vice-president, Mr. Merritt was made temporary chairman.

The secretary reported to the meeting the result of the mail ballot which was held during May, 1912, in order to determine the sentiment of the members of the society in regard to the publication of a journal and the transfer of the control of the *Physical Review* to the Physical Society. This mail ballot was ordered by the council after the close of the Boston meeting, April 27, 1912, the council having adopted the suggestion contained in the following letter from the editors of the *PHYSICAL REVIEW*:

To the Members of the Council of the American Physical Society:

In March, 1911, the editors of the *Physical Review* submitted a statement to the Council of the Physical Society with reference to the transfer of the *Review* to the Society. (This statement will be found on page 593 of the *Review* for June, 1911.) Since that time the Society has had the matter under consideration and has requested the Council to prepare a plan for publishing a journal and to determine under what conditions the control of the *Review* may be transferred to the Society. The fact that the members of the Council are so widely separated has, however, made it impracticable to hold a well attended meeting of the Council to discuss the details of the proposed transfer. In order to avoid further delay the editors of the *Review* suggest that the question be submitted to the members of the Society for a mail vote, with the understanding that if the Society votes to assume control of the *Review*, the present editors and advisory editors shall constitute the editorial board for the first year, and that a detailed plan for the permanent conduct of the *Review* be submitted by this board for the approval of the Society within a year of the date of transfer. The advisory editors of the *Review*, whose names now appear on the front inside cover, are J. S. Ames, K. E. Guthe, J. C. McLennan, W. F. Magie, R. A. Millikan, E. F. Nichols, B. O. Peirce, C. A. Skinner, John Zeleny.

If the Society decides that this transfer is desirable and is ready to assume the responsibility for the maintenance of the Review under the conditions named a year ago, the editors of the Review are ready to make the transfer at the earliest date that is found practicable.

If, however, the Society does not reach such a conclusion and is not ready to assume such responsibility, the editors desire, on the first of January, 1913, to definitely adopt the plan of a representative editorial board to coöperate with the present editors. They will ask the present advisory editors to constitute such a board in the first instance and to assist the editors in perfecting a permanent plan of operation.

February 22, 1912.

EDW. L. NICHOLS,
ERNEST MERRITT,
FREDERICK BEDELL.

The secretary reported that ballots had been sent to all regular members, and that those who failed to vote at once received a second ballot and a communication from the secretary calling attention to the desirability of the vote being a large and representative one.

The total number of regular members entitled to vote was 316. 201 votes were cast. The five propositions submitted to the society are given below with the number of votes in favor of each proposition.

1. That no change be made in the relation of the Society with the Physical Review. 87 votes.
2. That on January 1, 1913, the Society take over the Physical Review under the conditions specified, and in the manner suggested, in the accompanying communication from the editors. 97 votes.
3. That the Society give up its connection with the Physical Review and undertake the publication of an independent journal. 16 votes.
4. That the Council of the American Physical Society be given full powers as the representative of the said Society to obtain a certificate of incorporation for the said Society under the laws of the state which shall be deemed by the Council most suitable. 129 votes.
5. That, if the Society votes to assume control of the Review, the present editors and advisory editors shall constitute the editorial board for the first year, and that this board shall be requested to prepare a detailed plan for the permanent conduct of the Review, to be submitted for the approval of the council and of the society within a year of the date of transfer. (The advisory editors of the Review, whose names now appear on the front inside cover, are J. S. Ames, K. E. Guthe, J. C. McLennan, W. F. Magie, R. A. Millikan, E. F. Nichols, B. O. Peirce, C. A. Skinner, John Zeleny.) 130 votes.

The secretary pointed out that the original proposition received from the editors of the Physical Review in March, 1911, called for the approval of a majority of the members of the society in order that the control of the Review and the responsibility for its maintenance should pass to the society. In

order that proposition 2 should carry, it would, therefore, have required the favorable votes of 159 members, whereas only 97 votes were received. The editors had raised no objection to the vote being taken of the regular members only, although they had intended that all members should have a voice in deciding a matter affecting all equally.

After some discussion of the financial aspects of the proposition for the society to undertake the publication of the Review, a motion was made and carried, requesting the Council to inquire into the practicability of obtaining the joint guarantee of a number of colleges, so as to make the financial position of the Society secure in case it should undertake such publication.

The following papers were presented:

The Reflection of Electrons. ALBERT W. HULL.

Theory of Contact Rectifiers. (By title.) LOUIS COHEN.

Variation of Electrical Resistance with Temperature. A. A. SOMERVILLE.

Analogue of the von Waltenhofen Phenomenon in the Joule and Wiedemann Magnetostrictive Effects in Nickel and Steel Rods. (By title.) S. R. WILLIAMS.

Conditions for Obtaining Rapid Light Fluctuations from Incandescent Wires. C. F. LORENZ.

Comparison of Small Electrostatic Capacities. J. C. HUBBARD and H. F. STIMSON.

The Spectral Luminosity Curve of the Average Eye. HERBERT E. IVES.

The Heat of Solution of Radium Emanation. (By title.) R. W. BOYLE.

Effect of Frequency on the Constants of an Oscillating Electrical System.

J. C. HUBBARD.

The Asymmetric Emission of Secondary Radiation. O. W. RICHARDSON.

Equilibrium Figures. A. C. CREHORE.

The Intrinsic Brightness of the Glow Worm. HERBERT E. IVES.

Adjourned at 1 P.M.

ERNEST MERRITT,
Secretary.

ANALOGUE OF THE VON WALTENHOFEN PHENOMENON IN THE JOULE AND WIEDEMANN MAGNETOSTRICTIVE EFFECTS IN NICKEL AND STEEL RODS.¹

BY S. R. WILLIAMS.

IT has been shown by Von Waltenhofen² that when a magnetic force is suddenly removed from an iron rod it leaves less residual magnetism than when gradually removed. A study has been made of the Joule and Wiedemann magnetostrictive effects in nickel and steel rods to see how these different magnetostrictive effects are influenced by a sudden or a gradual removal of the field previous to the magnetic force applied for giving the various magnetostrictive effects.

¹ Abstract of a paper presented at the New York meeting of the Physical Society, Oct. 12, 1912.

² Von Waltenhofen, Wien. Ber., 48, (2), p. 564, 1863; Pogg. Ann., Vol. 120, 1863.

The results show that the magnetostrictive effects are different when the previous field is removed suddenly than when decreased slowly. From the standpoint of the planetesimal hypothesis of magnetism this is in agreement with predictions and the results are of value in helping to form some idea of what happens when a piece of ferromagnetic substance is subjected to a magnetizing force.

PHYSICAL LABORATORY,
OBERLIN COLLEGE,
OBERLIN, OHIO.

THE REFLECTION OF ELECTRONS.¹

By A. W. HULL.

VON BAEYER found that when electrons moving with velocities greater than 1.6×10^8 cm./sec. fall on a polished metal plate, more than 60 per cent. of them are reflected, and it has been generally assumed that for slower electrons the amount of reflection would be still greater. I find, however, that for the slow electrons produced by ultraviolet light, in an enclosure devoid of electric force, the amount of reflection is extremely small, certainly less than 1 per cent.

Two plates, 4 cm. in diameter, were mounted parallel to each other in a glass tube. One, of aluminium, was connected to an electrometer and illuminated at its center by a very narrow, carefully diaphragmed beam of ultraviolet light. The other, of silver, was movable along the tube, so that its distance from the first plate could be varied from 4 mm. (in which position no light fell upon it) to 15 cm. The inside of the tube was lined with blackened copper gauze. The dimensions of the plates were such that when they were 4 mm. apart practically all the electrons shot off from the illuminated plate would strike the other; and if any were reflected they would all return to the illuminated plate, thus diminishing the total current from it. When the plates were 15 cm. apart none of the reflected electrons could reach the illuminated plate.

I found the current from the illuminated plate to be the same when the plates were 4 mm. apart as when they were 15 cm. apart (and for all intermediate positions of the second plate), showing no effect of reflection.

To avoid any complication due to contact difference of potential the whole cavity, including gauze and aluminium plate, was sputtered with a cathode deposit of silver from the silver plate, and the potential difference between all three was tested by the Kelvin capacity method and found to be less than .1 volt. The experiment was then repeated, with the same result.

More extensive experiments are now in progress to test the connection between these results and those of Von Baeyer.

WORCESTER POLYTECHNIC INSTITUTE.

¹ Abstract of a paper presented at the New York meeting of the Physical Society, Oct. 12, 1912.

VARIATION OF ELECTRICAL RESISTANCE WITH TEMPERATURE.¹

BY A. A. SOMERVILLE.

THE materials tested were in the form of powder. A porcelain tube was filled with powder and nickel terminals inserted into either end of the tube, by which terminals a slight pressure could be placed on the powder. The terminals were also used as leads from the unknown resistance to the wheatstone bridge.

The resistance of all the materials tested decreases with an increase of temperature. The resistance given is for a column of powder one centimeter in diameter and about three centimeters in length. Below is given the temperatures at which such materials tested measure 10,000,000 ohms. In most cases during heating, the material undergoes a chemical change, uniting with oxygen of the air.

	Degrees Centigrade.
CaO.....	465
ThO ₂	700
U ₂ O ₃	340
Sb ₂ O ₃	470
Sb ₂ O ₄	430
Ni ₂ O ₃	300
SnO ₂	270
CuSO ₄	525
Fe ₂ SO ₄	375
NaCl.....	300
Alberene.....	800

Uranium trioxide conducts slightly at room temperature but at a slight increase in temperature is changed to U₂O₃ which is a non-conductor at ordinary temperatures.

Antimony trioxide when heated to a temperature of about 600 degrees is changed to antimony tetroxide. At 700 degrees it melts.

Green nickel oxide is converted into nickel sesquioxide, Ni₂O₃ at about 120 degrees.

At about 865 degrees copper sulphate is decomposed into copper oxide and sulphur trioxide.

The iron sulphate decomposes at 730 degrees.

If a small amount of borax is added to zinc oxide there is a sharp break in the temperature-resistance curve at 615 degrees where the borax fuses.

At 700 degrees sodium chloride begins to decompose liberating chlorine and forming sodium oxide, Na₂O.

THE SPECTRAL LUMINOSITY CURVE OF THE AVERAGE EYE.¹

BY HERBERT E. IVES.

PREVIOUS papers of this series² record an extended investigation of methods of heterochromatic photometry. As a consequence of this

¹ Abstract of a paper presented at the New York meeting of the Physical Society, Oct. 12, 1912.

² PHYSICAL REVIEW, XXXII., p. 441; XXXIII., p. 561; XXXIV., p. 387; XXXIV., p. 389.

investigation a set of conditions for colored light photometry have been suggested for adoption, as follows:

1. The use of the flicker photometer.
2. An illumination of 25 meter candles.
3. A photometric field of 2° diameter surrounded by an approximately equally bright field of 25° diameter.
4. The use of a normal or average eye.

The fourth requirement demands a knowledge of the characteristics of the average eye. To meet this need spectral luminosity curves of eighteen observers have been determined by the method and conditions mentioned. The average value which may be considered as taken of a sufficiently large number of observers to be adopted as standard is given in the following table.

LUMINOSITY CURVE OF AN EQUAL ENERGY SPECTRUM.

<i>Mean of Eighteen Observers.</i>	
.44 μ	.029
.45	.047
.46	.073
.47	.107
.48	.154
.49	.253
.50	.363
.51	.596
.52	.794
.53	.912
.54	.977
.55	1.000
.56	.990
.57	.948
.58	.875
.59	.763
.60	.635
.61	.509
.62	.387
.63	.272
.64	.175
.65	.104
.66	.068
.67	.044
.68	.026

CONDITIONS FOR OBTAINING RAPID LIGHT FLUCTUATIONS FROM INCANDESCENT WIRES.²

BY C. F. LORENZ.

THE convenience and reliability of incandescent filaments compared with arcs and sparks make it worth while to consider under what conditions their use is practicable in cases when an illuminant is required which will re-

¹ Extrapolated.

² Abstract of a paper presented at the New York meeting of the Physical Society, Oct. 12, 1912.

produce as light fluctuations any fluctuations in the electrical energy supplied to it. Compared with the spark the frequencies at which such filaments may be operated will of course be low.

Since the factor which retards and smooths out the temperature fluctuations is the thermal capacity of the filament, the first step toward making such an illuminant useful as a source of light for stroboscopic work is to choose the diameter of the filament as small as possible, since reducing the diameter reduces the thermal capacity in a greater ratio than the radiation. Assuming that the diameter has been reduced as far as practicable the following methods of further increasing the stroboscopic effectiveness suggest themselves: Running the filament at excessive temperature to increase the rate of cooling; using a color screen to limit the transmitted light to the shorter wave lengths, for which the fluctuation is greater than for the longer wave lengths; operating the filament in a cooling gas. Experiment shows that of these the last is by far the most effective. The ratio of the energy abstracted by the gas to the energy radiated, although not great for comparatively large wires (say of the order of 1 mm.) when they are at high temperatures, becomes very great for very fine filaments. The method involves a large waste of energy, but this may in some cases be of no moment, as for example in the measurement of alternating current frequency.

When operated with alternating current of any ordinary frequency a very fine filament lamp, say a 10 watt, 110 volt, tungsten lamp, gives only a very blurred stroboscopic effect, whereas even a larger filament, say one from a 25 watt, 110 volt, tungsten lamp, when mounted in an atmosphere of ammonia gives an effect amply sharp for frequency measurement of such currents. While the energy fluctuation is approximately a sine-squared function of the time, the light fluctuation is much more abrupt, the light being quite discontinuous, which is of course explained by the fact that the light emitted by a body varies much more rapidly than its temperature. In ordinary vacuum incandescent lamps on the other hand, the light fluctuations under alternating current operation are pretty closely sinusoidal. If, instead of the sinusoidal alternating current we use an intermittent current having energy concentrated in sharp pulses, the stroboscopic sharpness obtainable from the wire operated in a gas is much increased.

THE ASYMMETRIC EMISSION OF SECONDARY RAYS.¹

BY O. W. RICHARDSON.

THE fact of the emission of more secondary radiation from thin plates in the emergent than in the incident direction is now well established in a number of different cases. The existence of these effects has been taken to indicate that the primary rays are either material (Bragg) or at least have a discontinuous or atomic structure. Since the effect is shown by light, the latter view, which has been advocated on these or other grounds by Sir. J. J. Thomson,

¹ Abstract of a paper presented at the New York meeting of the Physical Society, Oct. 12, 1912.

Stark and other writers, has probably received more favor. The object of the present paper is to see if an estimate of the maximum effects to be observed may not be obtained without making any definite hypothesis about the structure of the radiation. So far as the author is aware there are no facts known at present which definitely exclude the possibility of an explanation of these effects on the basis of the usual continuous electromagnetic theory of light and X radiations. The paper is practically confined to the case in which the secondary radiation consists of emitted electrons.

It is assumed that the action between the radiation and the matter is primarily one between radiation and electrons, and that this action, in the case of any particular electron, is a continuous one until disruption occurs. The act of disruption terminates the continuity of the preceding processes and is succeeded by a new cycle of similar events. In general disruption only involves the breaking up of a previously existing dynamical system, but in favorable cases it may result in electronic emission. The condition for disruption in the cases considered is that the energy of an electron should be equal to $h\nu$, where ν is the frequency of the radiation and $h = 6.55 \times 10^{-27}$ erg sec. An estimate of the maximum degree of asymmetry to be found in the most favorable cases is obtained by assuming that up to the instant of disruption, no part, either of the energy or of the momentum, which any particular electron receives from the radiation, is communicated to the rest of the matter.

Consider the case of X radiation or light incident normally on a thin plate of absorbing matter. In the steady state it follows from the principle of the conservation of energy that the energy lost by the radiation in any interval is exactly equal to the energy $Nh\nu$, at the instant of disruption, of the N electrons disrupted during that interval. And further, if the principle of the conservation of momentum is applied to the whole system of matter and radiation, the momentum lost by the radiation is equal under the same conditions to the sum of the momenta of the N disrupted electrons. If $\bar{u}$ is the average velocity of them in the direction of the primary radiation, m their mass, and λ the momentum lost by the incident radiation when it loses unit energy, then

$$\bar{u} = \frac{\lambda h\nu}{m} = \frac{\lambda}{2} v^2, \quad (1)$$

where v^2 is the mean square of their velocity at the instant of disruption. On the electromagnetic theory $\lambda = 1/c$ and

$$\bar{u} = \frac{h\nu}{cm} = \frac{v^2}{2c}. \quad (2)$$

These relations are not sufficient to determine the distribution of velocity among the particles. But the asymmetry should be more marked the greater $\bar{u}/v$ and hence the greater the value of v . Thus those radiations which give rise to the emission of electrons with the greatest velocities should exhibit the greatest degree of asymmetry in this emission. This is known to be the case. As an illustration the limiting value $\bar{u}/v = 1/2$ for $v = c$, the velocity of light, is considered.

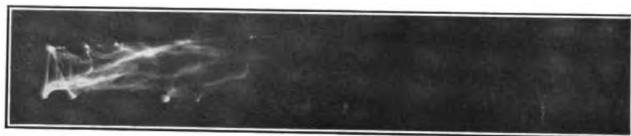


Fig. 2.

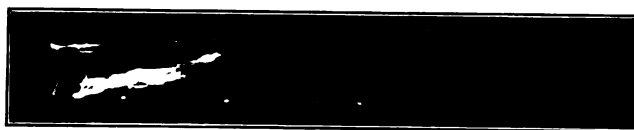


Fig. 3.



Fig. 4.

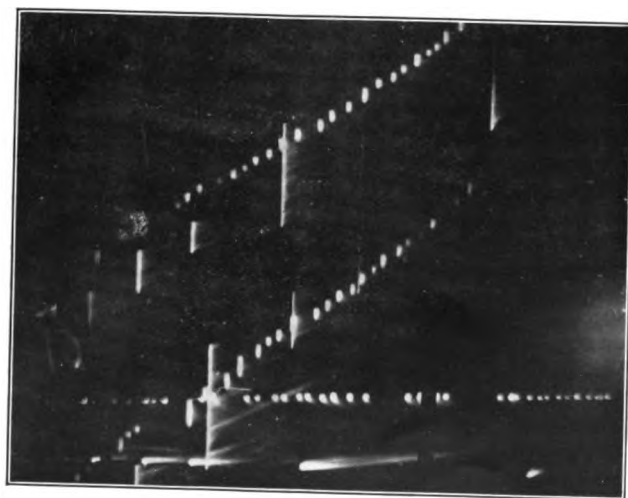


Fig. 5.

R. R. RAMSEY.

In the neighborhood of $\nu = 10^{15} \text{ sec}^{-1}$, u/ν from (2) is only about 1/500. The asymmetry found by Stuhlmann with ultraviolet light seems greater than this value of u/ν would lead us to expect. However, the case of ultraviolet light is different from that of the other radiations, since we are near the limiting value of ν below which no emission occurs. A small asymmetry is shown to be equivalent to a proportional change in the length of the effective spectrum and thus to produce a magnification of the effect. Under certain circumstances the experimentally observed asymmetry should be greater with the more electro-negative elements. There is some evidence in favor of this in the experimental results.

In the case of the asymmetric emission of radiation of the *X* type also, the explanation offered by the electromagnetic theory, along the line considered by Sommerfeld, is still one which has to be reckoned with.

PALMER PHYSICAL LABORATORY,
PRINCETON, N. J.

THE OSCILLATORY DISCHARGE OF A LEYDEN JAR.¹

By R. R. RAMSEY.

WHILE setting up the apparatus to show the oscillatory discharge of a Leyden jar by the revolving mirror method I noticed that the mirror fanned the spark to one side of the terminals of the spark gap and thus spread the discharge. The thought occurred to me that an air blast might be used to show the oscillation. Two iron rods were placed one above the other in a wooden support, the divergence from parallelism being very slight and the distance at the closer end being approximately one centimeter. A blast of air was directed against the vertex of the triangle thus formed. These rods were then made the spark gap of the oscillatory circuit. The discharge which originates between the rods at the closest point follows the ionized air as it is carried horizontally by the air current. Thus the spark is spread over a space of five or ten centimeters. When the condenser consists of five or six ordinary leyden jars and is charged by a medium-sized glass plate machine the discharge takes place once every five or ten seconds.

As a lecture experiment this has the advantage over the revolving mirror method in that every discharge is visible to every member of the class irrespective of his position in the room. In looking up the literature on the subject I find that this method has been used by L. Zender (*Ann. der Phys.*, 314, 1902, p. 899) and by G. A. Hemsalech (*Compt. Rend.*, 140, 1905, p. 1103). They give photographs of the discharge but the detail of the discharge is lost in the photograph due to the fact that several discharges are superimposed upon the plate.

The discharge was so beautiful that it was thought worth while to photograph it. In order to increase the brightness of the spark the terminals were

¹ Abstract of a paper presented at the Chicago meeting of the Physical Society, December 2, 1911.

made of magnesium ribbon mounted between wooden strips, Fig. 1. By this means a single discharge made a good impression on the plate.

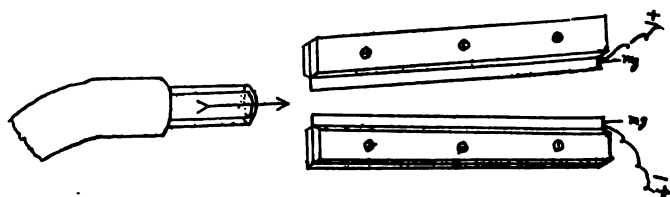


Fig. 1.

Fig. 2 shows the discharge when the capacity, C , is .012 M.F. and the inductance, L , is 65 henrys.

Fig. 3, $C = .025$ M.F. and $L = 65$ henrys.

Fig. 4, $C = .05$ M.F. and $L = 65$ henrys.

As the capacity is increased the discharge takes on more of the character of a flame and the detail of the discharge between the terminals is lost, but the discharge points on the terminals become more luminous, as is shown in Fig. 4.

In order to show better the character of the discharge a sensitive film was wound around an eight-inch motor pulley and placed so that the focal plane of the camera was tangent to the film.

Fig. 5 shows a horizontal line of discharge taken when the motor was not running, and two lines of discharge at about 45 degrees taken, when the motor was running at 1,200 R.P.M.

This shows that each oscillation occurs at regular intervals along the terminal.

The inductance used was the secondary of an eight-inch induction coil of 12,000 ohms resistance and an inductance measured by means of a standard inductance and a sechometer of 65 henrys without the iron core in place and 140 henrys with the iron in the core. The capacity was a variable glass plate condenser imbedded in paraffin. The values given are those obtained by ordinary laboratory methods.

INDIANA UNIVERSITY,

July 17, 1912.

THE PHYSICAL REVIEW.

VISCOSITY AND FLUIDITY¹—A SUMMARY OF RESULTS.² I.

BY EUGENE C. BINGHAM.

I. THEORETICAL RELATION BETWEEN VISCOSITY AND FLUIDITY.

THE coefficient of fluidity is measured by the velocity given to either of two horizontal planes, at unit distance apart, in respect to the other, by a unit tangential force per unit area, the space between the planes being filled with the viscous substance. But it is also the reciprocal of the well-known coefficient of viscosity, *i. e.*, $\varphi = 1/\eta$. Vis-

¹ Fifteenth communication bearing upon this subject.

² 1. Bingham, The Conductivity and Viscosity of Solutions of Certain Salts in Mixtures of Acetone with Methyl Alcohol, with Ethyl Alcohol, and Water. Dissertation, Johns Hopkins University, 1905.

2. Jones and Bingham, *do.*, Amer. Chem. J., 34, 481 (1905).

3. Bingham, Viscosity and Fluidity—Liquid Mixtures. Amer. Chem. J., 35, 195 (1906).

4. Jones, Bingham, and McMaster, "Über Leitfähigkeit und innere Reibung von Lösungen gewisser Salze in den Lösungsmittelgemischen: Wasser, Methylalkohol, Äthylalkohol, und Aceton," *Zeitschr. f. physik. Chem.*, 57, 193 (1906).

5. Jones, Lindsay, Carroll, Bassett, Bingham, Rouiller, McMaster, and Veazey, Conductivity and Viscosity in Mixed Solvents. Carnegie Institution of Washington Publication No. 80 (1907).

6. Bingham, "Viscosity and Fluidity" (Preliminary Paper), Amer. Chem. J., 40, 277 (1908).

7. Bingham, "Algebraische Analyse der Viskositätsdaten," *Zeitschr. f. physik. Chem.* 66, 238 (1909).

8. Bingham and Miss Harrison, "Viskosität und Fluidität," *Zeitschr. f. physik. Chem.*, 66, 1 (1909).

9. Bingham, "Viscosity and Fluidity," Amer. Chem. J., 43, 287 (1910).

10. Bingham, "Viscosity and Fluidity of Matter in the Three States of Aggregation and the Molecular Weight of Solids," Amer. Chem. J., 45, 264 (1911).

11. Bingham and White, "The Viscosity and Fluidity of Emulsions, Crystalline Liquids and Colloidal Solutions," J. Amer. Chem. Soc., 33, 1257 (1911).

12. Bingham and Durham, "The Viscosity and Fluidity of Suspensions of Finely-Divided Solids in Liquids," Amer. Chem. J., 46, 278 (1911).

13. Bingham, "Fluidity and Vapor Pressure," Amer. Chem. J., 47, 185 (1912).

14. Bingham and White, "Fluidität und die Hydrattheorie. I. Die Viskosität von Wasser," *Zeitschr. f. physik. Chem.*, 80, 670 (1912).

cosity and fluidity are analogous to electrical resistance and conductance respectively, the Law of Poiseuille,

$$\frac{8l}{\pi r^4} \cdot v = \frac{p}{\eta}$$

being the analogue of Ohm's Law,

$$C = \frac{E}{R}$$

since $8l/\pi r^4$ may be made constant, where v is the velocity of transpiration, p the pressure, η the viscosity, C the current, E the electromotive force, and R the resistance.

Case I. Fluidities are Additive.—Let Fig. 1 represent a section through a series of parallel lamellæ. If the surface AB is maintained at a constant difference of potential above CD , the total current C which will flow will be the sum of the partial currents flowing in the different lamellæ. Thus

$$C = [c_a + c_b + \dots] = \frac{E}{r_a} + \frac{E}{r_b} + \dots = \frac{E}{R},$$

where $r_a, r_b, \dots$ represent the resistances of their respective lamellæ $a, b, \dots$ and R the total resistance between AC and BD . So if AC is sheared downward with respect to BD , the resulting viscous flow Q is the sum of the amounts of flow in each lamella:

$$Q = [q_a + q_b + \dots] = \frac{p}{\eta_a} + \frac{p}{\eta_b} + \dots = \frac{p}{H},$$

where $\eta_a, \eta_b, \dots$ represent the viscous resistances of their respective lamellæ and H the total viscous resistance between AC and BD . Therefore

$$\frac{1}{H} = \frac{1}{\eta_a} + \frac{1}{\eta_b} + \dots$$

or

$$\Phi = \varphi_a + \varphi_b + \dots$$

and the coefficient of fluidity will be

$$\varphi = m_1\varphi_1 + m_2\varphi_2 + \dots, \quad (1)$$

where $m_1, m_2, \dots$ represent the fractions of the total volume occupied by the respective components present whose coefficients of fluidity are $\varphi_1, \varphi_2, \dots$. As a convention we will suppose the components to be arranged in the order of increasing fluidities.

*Case II. Viscosities are Additive.*¹—If the surface CA is at a constant difference of potential above BD , the resistance to the flow of the current will be the sum of the resistances of the different lamellæ

$$C = \frac{E}{r_a + r_b + \dots} = \frac{E}{R}.$$

Or if we imagine the lamellæ capable of viscous flow and that the surface AB is sheared toward the right, the resistance to flow will be the sum of the separate resistances

$$Q = \frac{p}{\eta_a + \eta_b + \dots} = \frac{p}{H'},$$

and therefore

$$H' = \eta_a + \eta_b + \dots$$

and the coefficient of viscosity will be

$$\eta' = m_1\eta_1 + m_2\eta_2 + \dots \quad (2)$$

The analogy is so nearly perfect that it is necessary to guard against one point of difference. In electrical flow, the conductors themselves remain stationary, but in viscous flow the substances themselves are

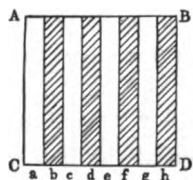


Fig. 1.

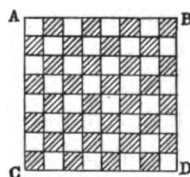


Fig. 2.

deformed. If the two above schemes are combined, a checkerboard arrangement (Fig. 2) is obtained and the electrical resistance of such a system of conductors is intermediate between those of the simple patterns. Thus one might be led to suppose that in mixtures of liquids, the viscous

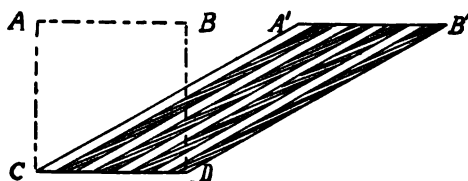


Fig. 3.

¹ Cp. eleventh paper.

resistance would always be intermediate between that given by the assumption that fluidities are additive and that given by the assumption that viscosities are additive. Such supposition is quite unwarranted, for if in Fig. 1 the surface *AB* is sheared to the right in respect to the surface *CD*, the individual lamellæ will tend to become inclined toward the right and very greatly elongated. An early stage of this development is depicted in Fig. 3. Thus although at the beginning the viscosities are strictly additive, as the viscous flow progresses, the conditions change more and more until the *fluidities* are additive. In miscible liquids, the condition that viscosities are additive appears to be very fleeting, disappearing as soon as the liquids are fully mixed. If the substances are immiscible, cohesion may prevent the continued elongation of the lamellæ, in which case the viscosities may be additive. In the case of emulsions, the conditions may be intermediate. This important case will be discussed later.

EXPERIMENTAL VERIFICATION OF THE THEORY IN LIQUID MIXTURES.¹

It now seems extraordinary that investigators have so long held to the belief that viscosities are under all conditions additive, particularly when the experimental evidence gave no support to their belief. To show that this is true, one needs only quote their own conclusions. Dunstan,² for example, says: "The law of mixtures is never accurately obeyed and divergences from it seem to be more clearly marked out in the case of viscosity than with other properties, such as refractive index." Thorpe and Rodger³ go deeper when they say: "The observations described in this paper afford additional evidence of the fact indicated by Wijkander⁴ and supported by Linebarger,⁵ that the viscosity of a mixture of miscible and chemically indifferent liquids is rarely, if ever, under all conditions, a linear function of the composition. It seldom happens that the liquid in a mixture preserves the particular viscosity it possesses in the unmixed condition. To judge from the instances heretofore studied, the viscosity of the mixture is, as a rule, uniformly lower than the mixture law would indicate, but no simple relation can yet be traced between the viscosity of a mixture and its chemical constituents." They are so confident that there is some uncomprehended principle which causes the viscosity curves to fall below the expected straight line, that they are dubious about some of the viscosity curves of Linebarger which

¹ Cp. third paper.

² J. Chem. Soc., 85, 819 (1904).

³ *Ibid.*, 71, 374 (1897).

⁴ Lunds, Phys. Sällsk. Jubelskrift, 1878; Wied. Beibl., 8, 3 (1879).

⁵ Am. J. Sci., 11, 331 (1896).

are very nearly straight lines. They say¹ "The observed viscosities in general are less than those calculated by the mixture rule, except, possibly, in the case of mixtures of carbon disulphide with benzene, toluene, ether and acetic ether, where, possibly, the temperature of observation (25°) was too near the boiling point of the carbon disulphide to make any specific influence, which that liquid might exert at lower temperatures, perceptible." That this assumption was without experimental justification, may be shown by the fact that these same investigators² had very carefully measured the viscosities of many liquids, including carbon disulphide, from zero to the boiling-point of each liquid without finding any peculiarity in the viscosity curve at the higher temperatures.

It may be shown that the viscosity of a mixture when the fluidities are additive will generally be smaller than when the viscosities are additive. Equations (1) and (2) represent these respective assumptions and for convenience we will assume that only two components are present. From equation (2) we get that

$$\frac{1}{\varphi'} = \frac{m_1}{\varphi_1} + \frac{m_2}{\varphi_2}$$

or

$$\varphi' = \frac{\varphi_1 \varphi_2}{m_2 \varphi_1 + m_1 \varphi_2}.$$

When $m_1 = 0$ or 1 , and $m_2 = 1$ or 0 respectively, φ must be equal to φ' . For all intermediate values of m_1 and m_2 we desire to learn whether φ must be invariably greater than, equal to, or less than φ' . Multiplying equation (1) by unity, we obtain

$$\begin{aligned} \varphi &= \frac{(m_1 \varphi_1 + m_2 \varphi_2)(m_2 \varphi_1 + m_1 \varphi_2)}{m_2 \varphi_1 + m_1 \varphi_2} \\ &= \frac{(m_1^2 + m_2^2) \varphi_1 \varphi_2 + m_1 m_2 (\varphi_1^2 + \varphi_2^2)}{m_2 \varphi_1 + m_1 \varphi_2} \gtrless \frac{\varphi_1 \varphi_2}{m_2 \varphi_1 + m_1 \varphi_2} = \varphi'. \end{aligned}$$

Since $m_2 = 1 - m_1$,

$$2m_1(m_1 - 1)\varphi_1\varphi_2 + m_1(1 - m_1)(\varphi_1^2 + \varphi_2^2) \gtrless 0.$$

Discarding the known roots, $m_1 = 0$ and $m_1 = 1$, we get

$$\varphi_1^2 - 2\varphi_1\varphi_2 + \varphi_2^2 \gtrless 0,$$

¹ J. Chem. Soc., 71, 361 (1897).

² Phil. Trans., 185A, 397 (1894).

which is a perfect square and therefore must be positive. Hence when φ_1 is equal to φ_2 , φ is equal to φ' , but at all other times φ must be greater than φ' . This conclusion may be stated as follows: *The viscosity of a thorough mixture of chemically indifferent liquids must always be less than would be expected on the assumption that viscosities are additive, but this inequality will approach zero as the difference between the viscosities of the components approaches zero. On the other hand, the viscosity of an emulsion (cp. later) must be greater than that of a perfect mixture of the same composition.* Thus is explained the irregular drop in fluidity as a mixture is cooled below its critical solution temperature, to be discussed later, as well as the peculiarities of perfect mixtures noted above.

Equation (1) may be expressed in the form

$$\varphi = \varphi_1 + (\varphi_2 - \varphi_1)m_2, \quad (3)$$

where φ_1 and φ_2 are constants and φ and m_2 are variables. The corresponding viscosity equation is

$$\eta = \frac{1}{\varphi_1 + (\varphi_2 - \varphi_1)m_2}. \quad (4)$$

It is very important to note that equation (3) is the equation of a straight line, but that equation (4) is the equation of a hyperbola. If we replace $\varphi_1 + (\varphi_2 - \varphi_1)m_1$ by $(\varphi_2 - \varphi_1)m$ and where

$$m = m_1 + \frac{\varphi_1}{\varphi_2 - \varphi_1}$$

we get

$$m\eta = \frac{1}{\varphi_2 - \varphi_1}$$

which is the equation of an equilateral hyperbola, whose X -axis is at a distance $\varphi_1/(\varphi_2 - \varphi_1)$ to the left of the origin to which equation (4) is referred. From equation (4) we find that the curvature is

$$k = \frac{2(\varphi_2 - \varphi_1)^2[\varphi_1 + (\varphi_2 - \varphi_1)m_1]^3}{\{[\varphi_1 + (\varphi_2 - \varphi_1)m_1]^4 + (\varphi_2 - \varphi_1)^2\}^{3/2}}. \quad (5)$$

By differentiating this curvature in respect to the concentration m_1 , and equating to zero, we find the concentration where the curvature is a maximum to be

$$m_1 = \frac{\sqrt{\varphi_2 - \varphi_1} - \varphi_1}{\varphi_2 - \varphi_1}. \quad (6)$$

Substituting this value in equation (5), the amount of curvature, where the curvature is a maximum, is found to be

$$k = \sqrt{\frac{\varphi_2 - \varphi_1}{2}}. \quad (7)$$

It follows, therefore, as a necessary consequence of the assumption that fluidities are additive that:

1. The curvature obtained by plotting viscosities is greatest when the difference between the fluidities, *i. e.*, $\varphi_2 - \varphi_1$, is large, and becomes zero when $\varphi_2 - \varphi_1 = 0$; *cp.* equation (7).

2. The curvature obtained by plotting viscosities must continually decrease as the concentration increases unless the square root of $\varphi_2 - \varphi_1$ is greater than φ_1 , in which case the point of greatest curvature will be found at some positive concentration [*cp.* equation (6)]. This latter case will hardly be met with except in the case of undercooled liquids and amorphous solids.

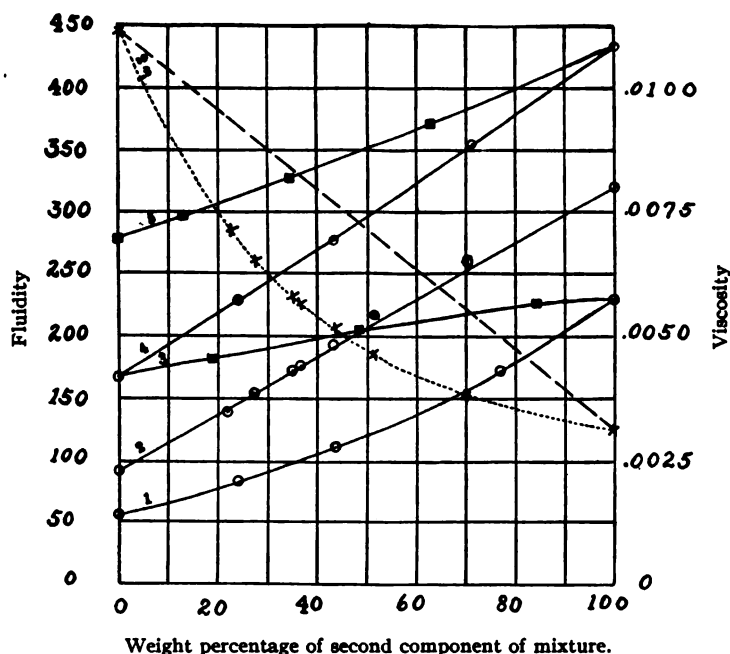


Fig. 4.

1, Fluidity curve of nitrobenzene and ethyl acetate at 25°; 2, fluidity curve of ethyl alcohol and acetone at 25°; 3, fluidity curve of benzene and ethyl acetate at 25°; 4, fluidity curve of benzene and ethyl ether at 25°; 5, fluidity curve of carbon bisulphide and ethyl ether at 25°; 2 η , viscosity curve (dotted) of ethyl alcohol and acetone at 25°. Were viscosities additive, this curve would be linear (dashes).

3. Mathematically considered, the curvature is dependent only upon the difference in the fluidities of the components, *i. e.*, $\varphi_2 - \varphi_1$ and not upon φ_1 [cp. equation (7)], but since we can only realize positive values of m_1 , it follows that for a given value of $\varphi_2 - \varphi_1$ the curvature at any concentration will be greatest when φ_1 is very small.

The truth of the first conclusion has been observed experimentally by Thorpe and Rodger¹ and White² and others.³ In reference to the second it has been noticed that viscosity curves often pass through a region of very great curvature, the "Erweichungsgebiet." This has been observed⁴ particularly of the changes of viscosity of a pure liquid with the temperature, but as will be pointed out later, these curves follow the same laws as do those of mixtures. The third conclusion is verified by the observations of White and others. However all of these peculiarities are characteristic of the viscosity curves only. The fluidity curves are far simpler and must therefore come to be used generally for purposes of physicochemical investigation. In Fig. 4 are given the fluidity curves of mixtures of a number of pairs of substances whose viscosity curves are particularly curved and the viscosity curve of one pair for comparison.⁵ It is seen that the fluidity curves are almost linear⁶ and hence are almost perfectly defined by their slopes and intercepts. It must be stated here that there are many mixtures which do not give linear fluidity curves, presumably because the components are not chemically indifferent to each other. These cases will be discussed in their appropriate place.

EXPERIMENTAL VERIFICATION OF THE THEORY IN PURE LIQUIDS AT VARYING TEMPERATURES.

It is evident that the reasoning which has been found to apply to mixtures of different liquids must also hold for mixtures of the same liquid at different temperatures; for a liquid at any temperature may be thought of as a mixture of appropriate amounts of portions of the liquid maintained at the extreme temperatures. Hence we are led to the hypothesis that *the temperature-fluidity curves of pure liquids should normally be linear, and the temperature-viscosity curves hyperbolic.* This

¹ J. Chem. Soc. (London), 71, 361 (1896).

² J. Indust. and Engin. Chem., 4, 270 (1912).

³ Cp. Bingham, Am. Chem. J., 35, 195 (1906).

⁴ Tammann, Z. physik. Chem., 28, 18 (1898).

⁵ Dunstan, J. Chem. Soc. (London), 85, 822 (1904); Linebarger, Am. J. Sci., 2, 331 (1896).

⁶ The concentrations were given in weight per cent. We have pointed out that volume percentages should have been used. That better results are obtained by using the later will be shown in connection with the discussion of hydrates, in our next paper.

may not experimentally be realized, for the volume of a liquid does not increase in a linear manner as the liquid is heated, and as will be indicated later, fluidity is apparently closely related to volume; and association and dissociation may play a disturbing effect. In Fig. 5 are given the

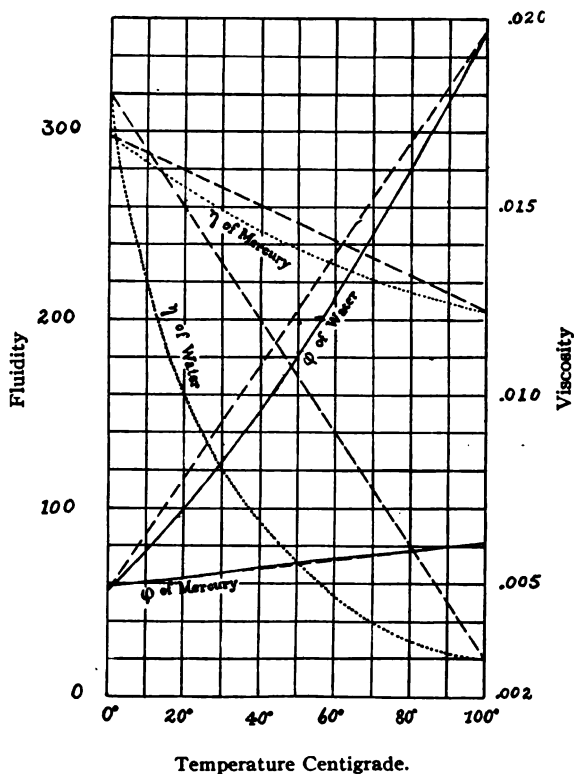


Fig. 5.

Fluidity (continuous) and viscosity (dotted) curves for mercury and water.

temperature-fluidity and temperature-viscosity curves of mercury and water from 1° to 100° C. Both of the former are much more nearly linear than the latter, the true linear curve being represented in each case by a series of dashes. The fluidity curve for mercury is almost absolutely linear, and what curvature there is is in a direction opposite to that of every other known substance, so that it can hardly be regarded as certain that this is not due to experimental error. The most marked deviations from linearity occur in the alcohols, which of course are highly associated, but even they approach linearity at high temperatures.

EXPERIMENTAL VERIFICATION OF THE THEORY IN EMULSIONS, SUSPENSIONS, AND COLLOIDAL SOLUTIONS.¹

Emulsions, suspensions, and colloidal solutions may profitably be considered together. Emulsions differ from the others in that both of the components have measurable fluidity. In suspensions and colloidal solutions the particles of suspended solid may be supposed to have zero fluidity. It has been shown above that if the individual drops of the emulsion are of sufficient size the viscosities may be additive and not the fluidities. A practical illustration of this is met with in stirring up a pail of "mixed paint," after the pigment has settled out on the bottom of the pail. The stirring is at first very difficult because the viscous resistance is that of the pigment deposit, which is very great, added to the viscous resistance of the oil which is small. Had the fluidities been additive in this case, the mixture would have stirred easily from the first, since adding the fluidity of the pigment to the very much greater fluidity of the supernatant oil would have given a resultant fluidity nearly the same as that of the oil alone. On thoroughly mixing the paint, the fluidity is raised (or the viscosity lowered) for the viscosities are no longer additive.

The hypothetical appearance of an emulsion as it flows through a capillary tube is represented in section in Fig. 6. Due to the friction



Fig. 6.

against the walls, the front end of the globules tend to become more convex and the rear end less convex than would be the case were the globules stationary. It is specially to be noted that when the globules are small in diameter and yet large enough to fill the whole cross-section of the tube, the motion of the liquid is not entirely linear, it being transverse as well as horizontal, as indicated by the arrows in the figure. Were the motion perfectly linear, we have already proved that the viscosities would be strictly additive, and that the resultant viscosity would be greater than would have been the case if the components had formed a perfect solution. But the effect of the transverse motion is to still further increase the viscosity. If, however, the drops are very large in comparison with the diameter of the tube, the effect of this transverse motion may be rendered negligible. It is self-evident therefore that if the globules of an emulsified liquid are large enough to fill the cross-

¹ Cp. eleventh and twelfth papers.

section of the transpiration tube, the viscosity will be at least as great as the sum of their viscosities multiplied by their respective volume concentrations [cp. equation (2)]. It may be greater due to the transverse motion. It may be remarked in passing that miscible liquids, when the mixing is excessively imperfect may show a resultant coefficient of viscosity which approaches the sum of the component viscosities; but it can never exceed that value because there can be then no transverse motion so long as the pressure is low enough to permit ordinary linear flow.

The above conclusion is very unambiguous, and mixtures of liquids which have a critical-solution temperature seemed best suited for the experimental confirmation because the values of the fluidity when the components formed an emulsion or a perfect solution could both then be determined at nearly the same temperature. On looking up the literature, it was found that Ostwald, and Stebutt,¹ Friedländer,² Scarpa,³ Rothmund,⁴ and Bose⁵ had all found abnormally large viscosities in lowering the temperature below the critical-solution temperature. But they had none of them perceived that this was to be expected from the nature of viscous flow, so that they proposed several different theories to account for the phenomenon. For a discussion of these theories, we need only refer to a paper by White and the present writer.⁶ In Fig. 7 we give the fluidity curves for phenol and water mixtures taken from their paper. The data were partly determined experimentally and partly taken from the work of Scarpa and recalculated to a fluidity basis. It is seen that there is always a tendency for the fluidity to fall off abnormally as the critical-solution temperature is approached. The results are irregular as are those of some of the earlier workers, and this is to be expected since the viscosity is to some extent at least dependent upon the size of the drops. It was noted by Bingham and White that the abnormality in the fluidity curves began before the critical-solution temperature was actually reached, *i. e.*, in the opalescent condition of the liquid. It becomes necessary for us therefore to point out that the decrease in fluidity does not depend upon the formation of actual drops but merely upon a sufficient lack of homogeneity in the liquid. The opalescence itself is, we believe, an indication of the lack of homogeneity,

¹ Lehrbuch der allgem. Chem., 2 Aufl., 2^a, 684, etc.

² Zeitschr. f. physik. Chem., 38, 385 (1901).

³ Cimento [5], 5, 117 (1903), and J. Chem. phys., 2, 447 (1904).

⁴ Zeitschr. f. physik. Chem., 63, 54 (1908).

⁵ Physik. Zeitschr. 8, 313, 347 (1907); 9, 769, 707 (1908); 10, 32, 230, 406 (1909); Zeitschr. f. Elektrochem., 8, 449 (1907).

⁶ J. Amer. Chem. Soc., 33, 1257 (1911).

and it is not very difficult to explain. At the critical-solution temperature the affinity of the molecules of phenol for other molecules of phenol added to that of molecules of water for other molecules of water becomes

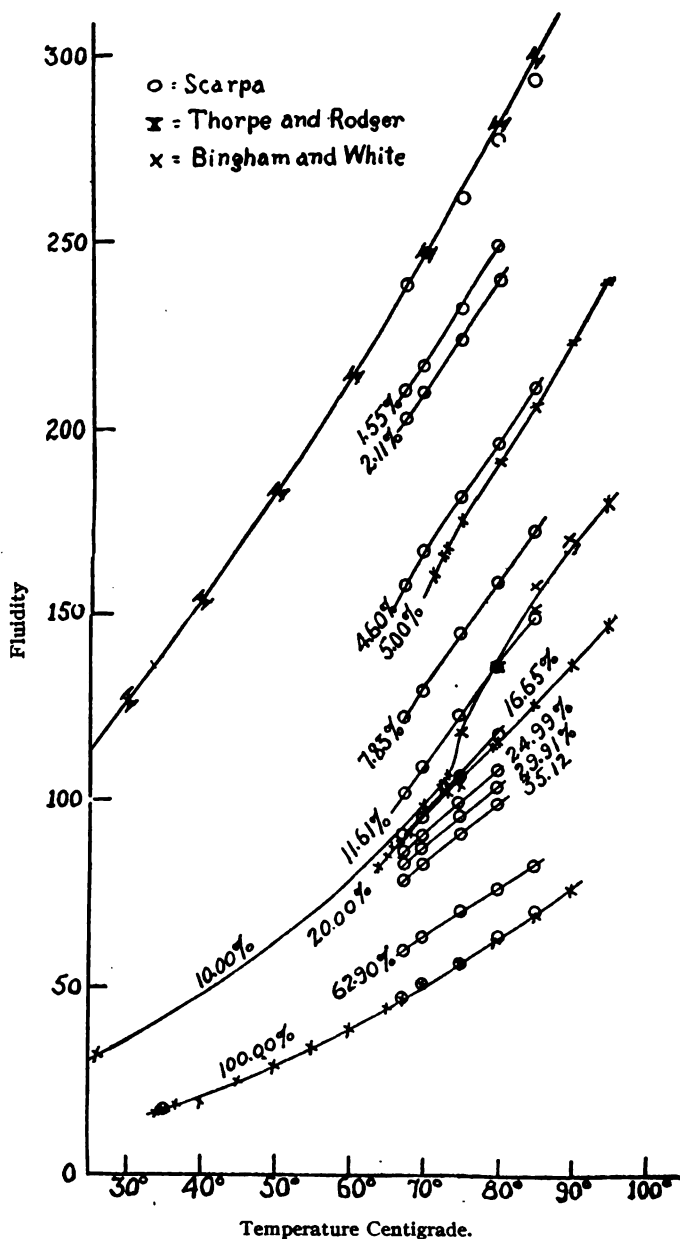


Fig. 7.

The fluidities of phenol and water mixtures.

equal to the affinity between the phenol and water. At the critical temperature, any slight changes in temperature have a maximum effect. But even at temperatures considerably above the critical-solution temperature, some of the phenol molecules and some of the water molecules must have velocities below the critical value. The result will be that even above the critical-solution temperature the molecules of phenol will group themselves together, and the molecules of water will likewise group themselves together, and if they reach sufficient size before being redissolved, the action of gravity will be appreciable in causing a separation of the two components.

It is evident that the same lack of homogeneity will explain the increase in viscosity in crystalline or "anisotropic" liquids. It is only necessary that the "swarms" be of sufficient size to make the viscosities additive.

We have seen that in perfect solutions the fluidities are normally additive. In emulsions the viscosities may be additive, the exact value depending upon the conditions of flow. When the suspended particles are very small as in most suspensions, it is not self-evident whether one or the other assumption is correct, or whether the value should be intermediate between the two. To clear up this point let us consider some simple cases.

If we imagine the solid particles of a suspension to be united into sheets parallel to the direction of flow, like the shaded areas in Fig. 1 when the surface CA is sheared upward in respect to DB , then it is evident that, provided that there is no slipping at the boundary between the liquid and solid, the flow will be the sum of the flows of the unshaded areas, *i. e.*, the fluidities will be strictly additive or

$$\varphi = \varphi_1 m_1 + \varphi_2 m_2 = \varphi_2 m_2, \quad (8)$$

since φ_1 is practically zero.

But if these solid sheets were broken into small fragments, one of which is shown as the cross-section of a cube at F in Fig. 8, the de-

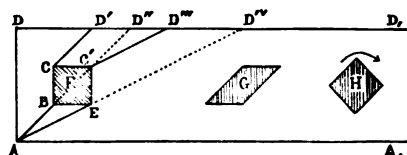


Fig. 8.

formation of the liquid would tend to change the cube into a parallelepiped shown at G , but as the solid is rigid, this cannot take place; so that the

shearing force can only rotate the cube around its center as shown at *H*. But the failure of the solid to change its shape with the flow of the liquid, will necessitate transverse motions in the liquid by way of readjustment, hence the viscosity of a suspension will always be greater than it would

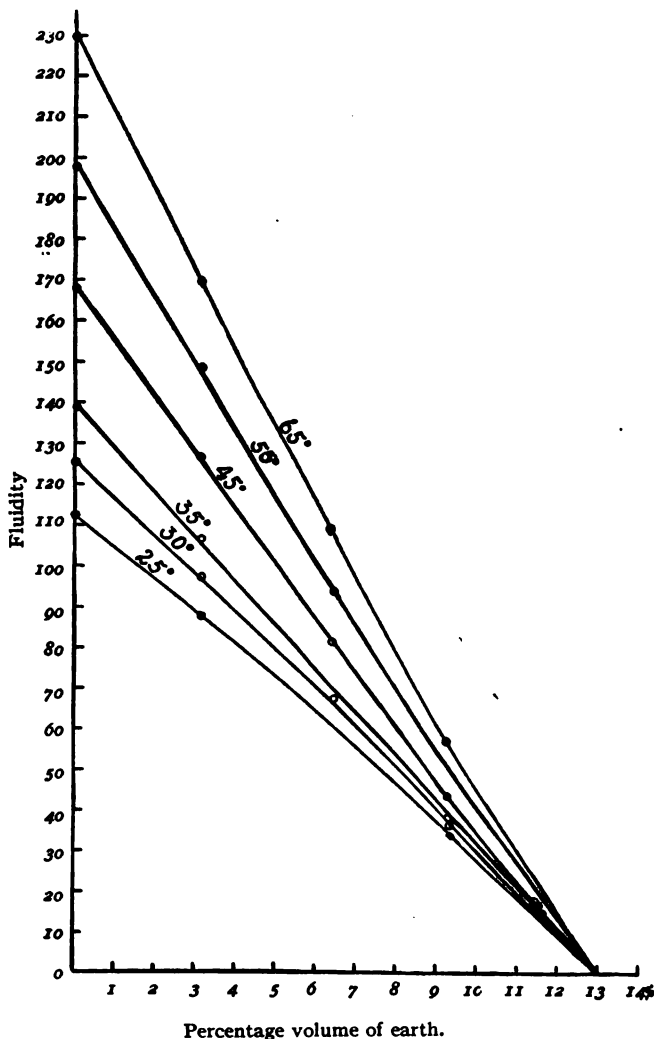


Fig. 9.

The fluidity of suspensions of infusorial earth in water at different temperatures and concentrations.

be were the fluidities strictly additive. This conclusion would not be true however if there were slipping at the boundary. Since if the viscosities were additive, the viscosity of a suspension would be infinite

which is certainly not always found to be the case, we reach the conclusion that *the fluidity of a suspension must always be intermediate between that calculated on either of the two fundamental assumptions, equations (1) and (2).*

A different way of looking at this question will be given later in discussing the behavior of a liquid at the boundary, which leads to the same conclusion. We will first give the fluidity curves of some suspensions of infusorial earth, china clay, and "Aquadag" in water and alcohol, as determined by Durham and the writer,¹ cp. Figs. 9-13. It is observed

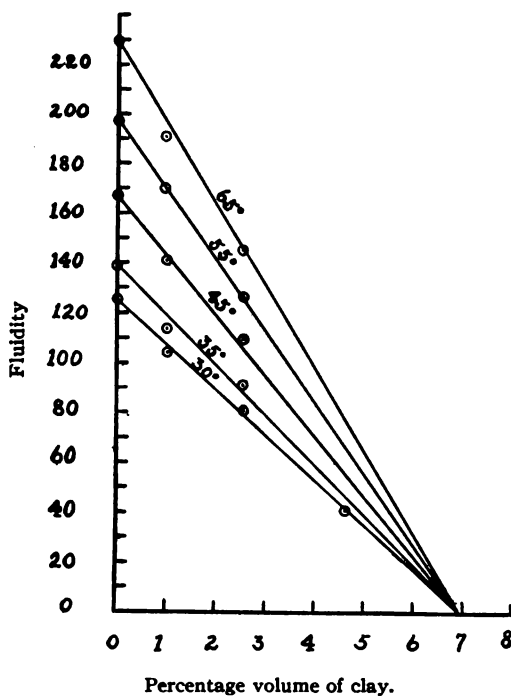


Fig. 10.

The fluidities of suspensions of English china clay in water at different temperatures and concentrations.

that *as the concentration of the solid present in a suspension rises, the fluidity of the suspension rapidly decreases and in a linear manner, so that at no very high concentration of solid, the fluidity would be zero, and even more extraordinary is the fact that this zero of fluidity is quite independent of the temperature.* Whether very greatly different pressures or other methods of viscosity measurement would affect these curves has not been

¹ Amer. Chem. J., 46, 278 (1911).

shown with certainty. It is of great importance that this point should be established.

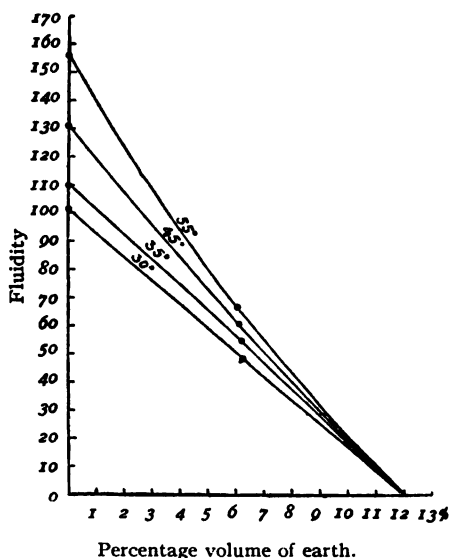


Fig. 11.

The fluidity of suspensions of infusorial earth in ethyl alcohol at different temperatures and concentrations.

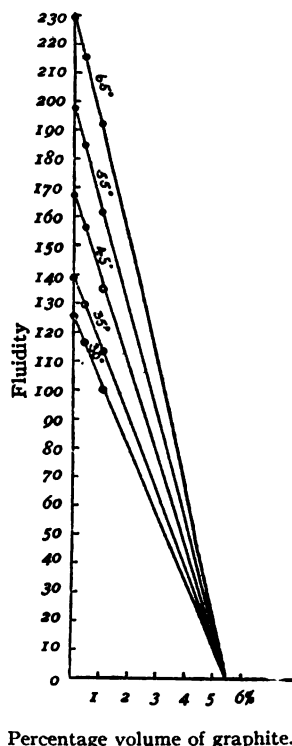


Fig. 12.

The fluidity of suspensions of graphite-Aquadag in water at different temperatures and concentrations.

This zero of fluidity in suspensions has apparently not heretofore been noticed. And it speaks very much in favor of the conceptions which we have been advancing that, with the old idea that only viscosities are additive, the remarkably simple relations which suspensions are here seen to exhibit, could scarcely have been discovered at all since the viscosity curves would only meet at infinity. But that these suspensions can have a zero of fluidity seems at first quite incomprehensible, because a suspension with zero fluidity is still sufficiently mobile so that if a bottle containing a considerable quantity of the mixture is inverted the material will run out. But *this* flow cannot be viscous flow, it must be plastic flow. This "zero of fluidity," we therefore believe to be the point of demarcation between viscous and plastic flow, the distinction being that in the former any shearing force, however small, will produce continuous

deformation, while in the latter a definite minimum of shearing force must be applied before continuous deformation will take place. If this explanation proves to be correct, it seems clear that the zero of fluidity ought to be dependent upon the pressure; for if sufficient pressure is at our command it seems probable that even the solid itself will show some degree of viscous flow. The fluidity formula of suspensions [cp. equation (8)] is

$$\varphi = \left(1 - \frac{m_1}{c_1}\right) \varphi_2, \quad (9)$$

where m_1 and m_2 are the volume concentrations of the suspended substance and the medium respectively, and c_1 and c_2 are their particular values where the fluidity becomes zero. Comparing this equation with equation (8) $\varphi = (1 - m_1)\varphi_2$ we see that, since c_1 is always less than unity, the effect of the transverse motions brought about by the presence of the solid particles is merely to increase the apparent concentration of the solid. It appears as it would if each particle of solid lowered the fluidity of the liquid immediately surrounding it to a negligible quantity, so that when the whole space is bridged over by these effects, the whole mixture may be said to possess "zero fluidity."

It will help in understanding this phenomenon to enquire into the behavior of a liquid near its boundary of solid. Helmholtz and Piotrowski¹ have given the results of experiments which indicate that slipping is considerable. The great prestige of Helmholtz made experiments which failed to show slipping carry less weight than would have been the case if the opposite conclusion had had a lesser champion. Hence a large number of somewhat elaborate experiments have been carried out by various experimenters² in order to decide the point. Ladenburg³ repeated with great care the actual experiments of Helmholtz and Piotrowski but without finding any evidence in favor of slipping. It therefore seems to be conclusively proved that perceptible slipping does not occur at the boundary between liquid and solid at least provided that they wet the solids over which they move.

And theoretically it seems most improbable that slipping should occur in any case at the boundary between a liquid and a solid, not only because of the inequalities in the ultimate surface of all solids, but because, with the high adhesion which is known to exist between all solids and liquids whether the solids are wet or not, it seems improbable that the flow in

¹ Wien. Ber., 40 (2a), 607 (1868).

² Whetham, Proc. Roy. Soc. (London), 48, 225 (1890); Hadamard, Compt. rend., 136, 299, 545 (1903).

³ Ann. Phys. [4], 27, 157 (1908).

the very thin layer in contact with the solid should be appreciable in comparison with the flow in the practically infinite number of layers of liquid which are moving over each other.

It is usually assumed that the layer of liquid in immediate contact with the solid is absolutely stationary, but it is much more logical to admit that there may be flow, and that it is subject to the well-known laws of viscous flow. The fundamental law of all linear viscous flow is that the rate of flow at any point in a homogeneous liquid in respect to a solid surface regarded as stationary is directly proportional to its distance from that surface. It is upon this fundamental law that all of our conceptions of viscosity ultimately depend. It tells us that velocities in a liquid increase in an arithmetical progression, so that the rate of flow near the boundary is of necessity very small, and in the flow of a liquid through a capillary tube it is well-known that a decrease in the diameter of only one half, causes a diminution in the volume of flow to one sixteenth of its former value. This result is so commonplace that its full effects have been generally overlooked. As an example it may be cited that lampwicking is woven so tightly that kerosene is not easily drawn up by the capillary forces which at these microscopic distances are very powerful. So the fabric is treated with steam which expands the pores. But with a less fluid substance like hot pitch or various gums the velocity of flow would still be negligible, nor can it be made to rise by still further enlarging the pores, because then the capillary attraction is greatly weakened. Hence when goods are to be saturated with a gum like rubber, a solvent is absolutely necessary in order to raise the fluidity to a workable quantity. Each of the solid particles of a suspension is, according to this explanation, surrounded by a layer of liquid which is practically stationary in reference to the particle—not absolutely so—and this gives rise to the zero of fluidity in a mixture with comparatively small percentage of solid present. As the number of particles in the suspension increased, this stationary space is also increased in proportion, *i. e.*, these stationary spaces do not overlap, because as two particles approach each other, they will not quickly penetrate each other's stationary spheres, when there is no strong specific attraction between the particles. In other words, the fluidity is lowered in a linear manner.

Thus may be explained why a suspension will often settle out quickly so as to give two distinct layers, but only after a very considerable time has elapsed will the layers of solid particles sink to their final volume, the particles approaching each other finally with infinite slowness. The same principle is involved in the clogging of a filter by means of fine

precipitates. The pores in the fine precipitates are smaller than in coarse-grained precipitates, and the smaller diameter of the pores very much more than makes up for their greater number, their area being the same in both cases as can easily be proved. Furthermore every chemist knows that if the suspended precipitate is poured on to the filter before wetting the paper, the trouble from clogging is increased. This is doubtless because the suspended particles are forcibly drawn into the pores of the paper by the powerful capillary forces, and then to make matters worse, the pores are partly closed up by the expansion of the cellulose on becoming wet.

Perhaps the most interesting question in regard to the fluidity of suspensions arises in regard to the effect of the size of the particles, for we have seen that the zero of fluidity is considerably different for suspensions of different materials. Since the substances used were insoluble in the liquids employed, one naturally looks for an explanation in the differences in the sizes of the particles. The explanation for the zero of fluidity which we have offered, would seem to demand that the fluidity decreases as the size of the particles is reduced, since thereby their surface is enlarged. This decrease in fluidity will however not continue indefinitely, because when the particles approach molecular dimensions, their vibrational velocity will become appreciable and in liquids the increase in vibrational velocity is associated with increased fluidity. Ordinary experience is in harmony with this inference for it tells us that the fluidity of a liquid is not greatly decreased by the presence in it of an amount of very coarse material, of nearly the same specific gravity as the liquid, which would very greatly decrease the fluidity if the particles were in a very fine state of subdivision. Aquadag must be very finely divided since it will stay suspended indefinitely, yet it depresses the fluidity more than any other suspension which has been studied by us. It is well known that very small amounts of colloids exert a very great effect in depressing the fluidity. The experiments of Woudstra¹ on the viscosity of colloidal silver solutions indicate that a zero of fluidity would be reached at a concentration of considerably less than one volume per cent. of colloidal silver.

Very fortunately for the views here presented, Woudstra has observed that colloidal silver solutions increase in fluidity on standing, and also on the addition of electrolytes. He recognizes that in either case coagulation is favored and he is thereby led to see that viscosity is here dependent upon the area of the boundary surface. He furthermore recognized that colloidal silver solutions are very different from colloidal

¹ *Zeitschr. f. physik. Chem.*, 63, 619 (1908).

solutions of gelatine and silicic acid, for with the latter the fluidity decreases with coagulation. Happily even this latter class of solutions does not speak against the views here presented as their behavior is readily explained. Many chemists believe that in a gelatine solution a "liquid precipitate" separates out as the coagulation proceeds, which forms a network through the liquid. If this is true, the fundamental law of viscous flow cannot be realized in that the rate of flow will not increase in direct proportion to the distance from the boundary of the mixture. The network of precipitate will extend the region of no flow out into the liquid, and the flow which takes place through the pores of the precipitate will be very slight. It is common experience that a gelatine solution after it has thoroughly set may be rendered considerably more fluid by simply cutting it in various directions or by grinding it in a mortar or by shaking. It is self-evident from what has been already said, that cutting the network of precipitate will increase the fluidity. Perhaps however there is no sharp line of demarcation between the fine compact precipitate of silver and the voluminous precipitate of gelatine.

Our discussion of the effect of the size of the particles upon the fluidity of the suspension would be incomplete, if we left the impression that the zero of fluidity is independent of the nature of the suspending liquid. In the curves already given, it is to be noticed that the zero of fluidity for infusorial earth is not quite the same in alcohol that it is in water. But far more extraordinary are the effects produced by the addition of small amounts of electrolytes to the suspensions in water, and contrary to the usual impression, the electrolytes are very diversified in their effects. For we have observed that the addition of small amounts of any substance known to give hydrogen ions invariably produces a drop in the fluidity of the suspensions with which we have worked. On the other hand, small amounts of substances known to give hydroxyl ions, cause an increase in fluidity, and salts which give a neutral reaction give a mixed effect. Moreover the addition of an acid to these suspensions produces marked coagulation. The coherence of the particles together in gelatinous masses causes the formation of networks throughout the liquid which probably lower the fluidity after the manner already explained for gelatine solutions.

It is not at all essential to our theory of the relation of fluidity and viscosity to each other and to other properties that we understand the cause of the increased coherence which is manifest in the presence of hydrogen ions, but the question is an interesting one and undoubtedly its complete answer would suggest further opportunities for testing our theory. Hence we shall state some of the reflections which our work up to the present time suggests.

The action of soap in cleansing has usually been attributed to the action of the hydroxyl ions in causing the saponification of the grease. But this hypothesis has always seemed inadequate and it fails entirely to explain the equally high efficiency of soap in removing mineral oils and greases. To say that the soap promotes the emulsifying of the oil or grease is a satisfactory restating of the phenomena but it is scarcely adequate as an explanation. The procedure of a workman in removing mineral grease from his hands is usually as follows. He first pours kerosene, gasoline, or other fluid oil upon his hands and mixes the grease and oil together by rubbing. Thus the fluidity of the grease is raised. With a little water and soap he then produces a thick lather on his hands; and incidentally, although vital to the success of the operation, the rubbing and consequent viscous flow involved in raising the lather loosens the grease, which has already been rendered fluid by means of the oil, and since the particles do not readily cohere in the alkaline solution they do not again become attached to the skin. To complete the operation it is therefore only necessary to wash off the lather. That *any* solution containing hydroxyl ions will be just as efficient as the soap solution with which it is isotonic cannot of course be maintained. But it is certainly true that many substances with an alkaline reaction, like borax, soda, and ammonia have marked cleansing power and that the addition of acid substances to soap solutions entirely destroys their cleansing power, the grease immediately re-attaching itself to the objects being cleaned. I am informed that a liquid soap is being used which consists essentially of an emulsion of a mineral oil in a dilute solution of caustic soda.

It appears that *emulsions and suspensions are generally more stable in alkaline than in acid solutions*. Many illustrations of this might be given, but a few must suffice. India ink is immediately coagulated by a trace of acid and the same is true of the suspension of carbon known as "Aquadag." Mercury can be easily emulsified, and it is customary to get it back again by boiling with hydrochloric acid. We have found that the Richmond deposits of infusorial earth may be elutriated and a portion of material obtained which will remain in suspension for many days. But this suspension is rapidly coagulated by an acid solution and the elutriation cannot be successfully carried out in acidulated water. In contrast with this, we have noted that English china clay, when stirred up in water, settles out quickly, leaving a clear supernatant liquid, while the material settling out is noticeably flocculent. On adding a little alkali, the coarser materials settle out first, and the liquid at the top becomes only very gradually clear. Moreover the flocculent

character of the deposit is destroyed, which is evidence of the loss of coherence between the particles. With the use of a ball mill clay may be brought into permanent suspension without the use of alkali. But the liquid becomes alkaline automatically on account of the hydrolysis of the feldspar as soon as the layer of insoluble aluminum silicate is removed by the mill from the surface of the particles. Every reader knows that when cream is sour, be it ever so slightly so, the coagulation is very evident as soon as the cream is poured into the morning cup of coffee. The writer is not at liberty to cite some important illustrations of this principle arising in the industrial arts. It is probably true that only the technical chemist, who habitually resorts to dilute alkali and a ball mill for getting substances into emulsion or suspension, can truly appreciate the generality of this principle.

Perhaps the illustrations given, together with others which will readily occur to the reader, will be sufficient to show that the coagulation in the presence of hydrogen ions is a somewhat general phenomenon, and that it is due to the increased coherence between the particles; but they are not sufficient to give a clear insight into the cause of this increased cohesion. Perhaps the following illustration will help to do this. When a solution of rosin in alcohol is thrown into water, a suspension is formed which is quite permanent since I have found that it would remain for many weeks without settling out. However on adding a drop of a dilute solution of a strong acid, the whole suspension is beautifully coagulated, clouds of flocculent precipitate floating about in the clear liquid. But on adding an equivalent amount of alkali the suspension is again formed nearly as at first, and on the addition of more the insoluble acid begins to dissolve. This change is so sharp and so readily detected that it is entirely possible to use such a suspension as an indicator in acidimetry and alkalimetry. It appears from this instance that the thinnest sort of a layer of the soluble salt, such as may occur in what is usually termed a neutral solution is sufficient to prevent the coherence of the particles. But as soon as a drop of acid is added this thin layer of salt is destroyed leaving a fresh surface of the insoluble acid. But while the particles of acid are not soluble in water they are soluble in each other,¹ hence cohesion according to the well-known phenomena of surface tension pulls the particles together and holds them. But as alkali is added the solid particles are again covered by a layer of solution which is soluble in water and perhaps at the same time in the free acid hence the particles are no longer held together. So the phenomena are to be explained by the difference in solubility between the alkali salt and the free acid. It

¹ Cp. article by the writer on "Solubility," Amer. Chem. J., 38, 91 (1907).

is true of abietic acid as of the greater number of organic acids that it is much more insoluble in water than are its alkali salts, hence the explanation of this case is capable of rather extended application. Of course it is not impossible according to this explanation that there may be cases where the addition of alkali will cause coagulation and a decrease in fluidity, but we do not happen to have discovered such. Perhaps we should consider a solution of ferric chloride to be an illustration of such a case, since here we have a colloidal solution of ferric hydroxide which is readily coagulated by the addition of alkali.

Having discussed the effect of the size of the particles in a suspension upon the fluidity of the mixture and also the extraordinary effect of small differences in the chemical character of the suspending medium, it becomes important to ask how the fluidity of the mixture is dependent upon the fluidity of the medium. To decide this, we may refer to Fig. 8, where the lower surface AA_1 is supposed to be stationary while the upper surface DD_1 is sheared to the right under a given force, such that the point D would be sheared to D'' in a unit time. The distance of any point in the line AD'' from the line AD represents the velocity in its respective layer, the motion being assumed to be low enough so that it is not turbulent. But if the layers from B to C are replaced by a continuous solid and if there is no slipping at the boundary, the velocity of flow will be constant from B to C and the curve representing the velocities will become $ABCD'$. There is a loss in velocity (or fluidity) of $CC' = D'D''$, which is proportional to the amount of liquid replaced by solid, so that

$$\frac{D'D''}{DD''} = \frac{BC}{AD} = m_1.$$

With another liquid, but the other conditions the same, the velocity at the surface DD_1 might be DD'' but the ratio BC/AD would be the same, so we reach the conclusion that *the reduction in the fluidity of a liquid by the addition of a given quantity of finely divided solid is directly proportional to the fluidity of the liquid* provided, of course, that solution or agglutination do not occur. This may also be deduced directly from equation (8). As we have seen, suspensions are somewhat more complicated than the above case on account of transverse motions, but the law applies to these transverse motions as well.

We have then a test of our theory which was for a long time overlooked by us. The law states that the reduction in the fluidity on adding infusorial earth to either alcohol or water must be in direct proportion to

their respective fluidities. Now at 55° C. the fluidities of alcohol and water are considerably different, being 156.5 and 197.8 respectively. But the theory demands that *the volume concentration of the mixture having zero fluidity should be the same in all media*. As a matter of fact the zero fluidity occurs at 12 volume per cent. in alcohol and 13 volume per cent. in water, which is satisfactorily close when we take into consideration the effects of small amounts of impurities and the different dissociating power of alcohol and water.

But the same medium has very different fluidities at different temperatures, and the theory demands that the reduction in fluidity will be in direct proportion to the fluidity which the medium possesses and therefore that *the volume concentration of the mixture corresponding to zero fluidity should be absolutely independent of the temperature*. This is observed experimentally to be the fact within the limits of experimental error.

CONCLUSIONS.

We have thus far given a summary of the theoretical and experimental evidence for the following conclusions:

1. The assumption, which seems to have been universally held, *viz.*, that viscosities are always additive, is quite erroneous.
2. Instead of viscosities being additive, it is much nearer the truth to state that fluidities are additive, which implies that viscosities are hyperbolic. But without careful definition this generalization is both insufficient and misleading. Therefore we add:
3. The fluidity of a thorough mixture of miscible and chemically indifferent liquids is a linear function of the volume-concentration, equation (1). Thus is explained the fact which has been repeatedly observed and not explained, that the viscosities of such mixtures are generally less than would be expected upon the current assumption.
4. The fluidity-temperature curve of pure liquids, or of thorough mixtures of miscible and chemically indifferent liquids, are normally linear. This conclusion is perfectly true for only a very few liquids, *e. g.*, mercury, but it is approximately true for all unassociated liquids and for associated liquids at high temperatures at least.
5. In special cases, the viscosities may be additive, as in either miscible or immiscible though chemically indifferent liquids, when the mixture by solution or emulsion respectively is very incomplete.
6. If the globules of liquid in an emulsion are small and well distributed throughout the mixture, the fluidity may apparently be either less (due to transverse motions) or greater (due to the approach to conditions

where the fluidities are additive) than would be the case if the viscosities were strictly additive (cp. 5), but the fluidity will always be less than it would be, if the fluidities were additive. This may be due to the cause given under 9.

7. In suspensions of finely-divided but insoluble solids, the fluidity will always be less than it would be if the fluidities of the components were strictly additive. As a matter of experimentally determined fact, the fluidity decreases in a linear manner so that at no very great volume concentration the mixture appears to have zero fluidity. The concentration of the mixture having zero fluidity appears to be independent of the nature of the solid or of the suspending medium provided that they do not result in the solution or agglutination of the solid particles; it is independent of the temperature of the medium; and it apparently depends solely upon the size and number of the suspended particles in a unit volume.

8. This mixture of zero fluidity apparently demarcates for the first time viscous from plastic flow.

9. Each particle of suspended solid behaves as if it not only possessed zero fluidity itself, but as if it is able to communicate this to a spherical shell of the liquid surrounding it, the thickness of this shell being apparently thicker at high temperatures than at low ones. This curious behavior is exactly what would be expected if there were no slipping at the boundary between solid and liquid, and is a consequence of the fundamental law of viscous deformation, that the velocity of flow at any point varies directly as the distance of the point from the boundary of the liquid. The greater the fluidity of the medium or what amounts, in general, to the same thing, the higher the temperature, the more pronounced is this effect.

10. As a further consequence of the above conclusion, it follows that the greater the area of solid surface exposed, either by increasing the concentration or by diminishing the size of the particles and thereby increasing their number, the greater will be the diminution in the fluidity of the medium. Suspensions of very fine matter and colloidal solutions afford satisfactory confirmation of this view.

11. In gelatinous precipitates and certain colloidal solutions like that of gelatine where the particles cohere together, forming a network, the solid boundary is projected out into the liquid, hence the fluidity of the mixture is exceedingly low.

12. The addition of electrolytes to suspensions and colloidal solutions produces marked effect upon the fluidity. When coagulation occurs, the fluidity will be increased on account of the reduction in the area of

boundary surface, but on the other hand the fluidity will be decreased if the coagulating particles cohere together in such a way as to form networks through the liquid; so the resultant effect may be either to increase or to decrease the fluidity.

13. In suspensions, acids generally causè gelatinous coagulation and a decrease of fluidity. Alkalies have the opposite effect and neutral salts have a mixed effect. This effect is due to the increased cohesion between the solid particles in a slightly acid solution; and this in turn is interpreted as due to the greater insolubility of most organic acids than of their alkali salts. The greater the insolubility of the acid in water, the greater will be its solubility in itself, *i. e.*, its cohesion.

This cohesion as well as some of the others is capable of considerable practical application. Thus the theory of the detergent action of soap is for the first time made thoroughly rational.

14. Colloidal silver solutions¹ show a decrease in fluidity on standing or on the addition of an electrolyte. This is interpreted as due to the reduction of surface by coagulation, the precipitated colloid being in this case probably crystalline and not gelatinous.

15. Further conclusions in regard to the character of viscosity curves in general have been given on page 413 and need not be repeated here.

The majority of workers still cling to the old hypothesis that viscosities are always additive, hence we have tried to make the argument as strong as possible to show that the hypothesis is untenable. It may be possible for the reader to give his assent to the general argument which is here advanced, and yet it has been true in the past that in applying it in isolated cases, the worker becomes easily confused because of the complexity of many of the actual cases. For example, almost all aqueous solutions give fluidity curves which are not linear. We hold however that this cannot be used to argue for either the truth or the falsity of the theory here proposed until we come to an understanding of the question of hydrates, which undoubtedly may, and probably does, here cause complication. In scrutinizing the evidence for or against any theory, it seems unthinkable that any evidence should be set aside except for a valid reason, and then it is to be set aside only temporarily, until the cause of the complication can be satisfactorily studied. At such a time, these excluded cases may be used as very important evidence for or against the theory.

In the second part of this paper, we shall regard the evidence already given for the theory of viscous flow here presented as conclusive; we shall therefore attempt to throw light upon the nature of the relations between

¹ Woudstra, *loc. cit.*

fluidity and other properties, using in particular those cases which have heretofore been excluded. If the reader feels that the evidence at any point is inconclusive, he is asked to note that since the evidence is cumulative he may well withhold judgment because the remaining part of this paper carries weighty evidence for or against the theory.

RICHMOND COLLEGE,
RICHMOND, VIRGINIA,
September 17, 1912.

DETERMINATION OF CAPACITIES BY MEANS OF CONJUGATE FUNCTIONS.

By J. W. WOODROW.

THIS paper is intended to make use of the properties of conjugate functions for the solution of a few problems. The cases to be considered are those in which the surfaces of electrical conductors are generated by the motion of straight lines all parallel to a straight line which will be taken as the axis of z . These conductors are to be considered sufficiently long in this direction so that when charged, the z -component of the electric force may be neglected. Then if that portion of the field between two planes parallel to the xy -plane and unit distance apart be considered, the potential and distribution of electricity become functions of x and y only. If a relation can be found between w and z where

$$w = u + iv \quad \text{and} \quad z = x + iy,$$

which will transform straight lines in the w -plane into curves in the z -plane which coincide with the lines of intersection between the equipotential surfaces and a plane perpendicular to them, expressions for the capacity of the system and for the surface distribution can be easily found.¹

In the following transformations, the equipotential surfaces will be given by u equal to a constant and the lines of force by v equal to a constant. The charge e will indicate the charge on unit length of the conductor and the capacity C will represent the capacity per unit length.

Consider first the simple case of the surface distribution on a very long wire parallel to an infinite plane conductor at zero potential. It is well known that the transformation

$$w = \log \frac{z + a}{z - a} \tag{1}$$

will give the equipotential lines and lines of force for the above condition. The capacity as given by this transformation will be found in Webster's *Electricity and Magnetism*, where it is also shown that the surface dis-

¹ *Electricity and Magnetism*, J. H. Jeans, p. 256; *Electricity and Magnetism*, A. G. Webster, p. 307.

tribution for any transformation by means of the complex variable is given by

$$\sigma = \frac{1}{4\pi} \left| \frac{dw}{dz} \right|. \quad (2)$$

By differentiating (1)

$$\frac{dw}{dz} = \frac{2a[(a^2 - x^2 + y^2) + 2ixy]}{(a^2 - x^2 + y^2)^2 + 4x^2y^2},$$

and hence

$$\left| \frac{dw}{dz} \right| = \frac{\sqrt{4a^2[(a^2 - x^2 + y^2)^2 + 4x^2y^2]}}{(a^2 - x^2 + y^2)^2 + 4x^2y^2}. \quad (3)$$

Whence

$$\sigma = \frac{a}{2\pi\sqrt{(a^2 - x^2 + y^2)^2 + 4x^2y^2}}. \quad (4)$$

Now a more convenient form is obtained by changing to polar coördinates with the origin at the center of the wire. If r is the radius of the wire and d is the distance of its center from the infinite plane which is represented by the y -axis, the following relation is easily deduced:

$$a^2 = d^2 - r^2.$$

Substituting this value of a in (4) and making the transformation to polar coördinates, the expression for the surface density becomes

$$\sigma = \frac{\sqrt{d^2 - r^2}}{4\pi r} \cdot \frac{1}{r \cos \varphi + d}. \quad (5)$$

This gives a minimum value of σ for $\varphi = 0$ and a maximum value for $\varphi = \pi$. The surface distribution on the infinite plane is obtained by putting $x = 0$ in (4) which then becomes

$$\sigma = \frac{a}{2\pi(a^2 + y^2)}. \quad (6)$$

WIRE PARALLEL TO TWO PLANES INTERSECTING AT RIGHT ANGLES.

The equipotential lines and lines of force about a long wire parallel to the intersection of two perpendicular infinite planes are given by the transformation

$$w = \log \frac{z^2 - \beta^2}{z^2 - \alpha^2}, \quad (7)$$

where

$$\alpha = a + bi \quad \text{and} \quad \beta = a - bi.$$

The curves in the z -plane for u equal to a constant are shown in Fig. 1. Near the point (a, b) the curves approximate very closely to circles so that the equipotential line C in Fig. 1 can be replaced by a wire of circular

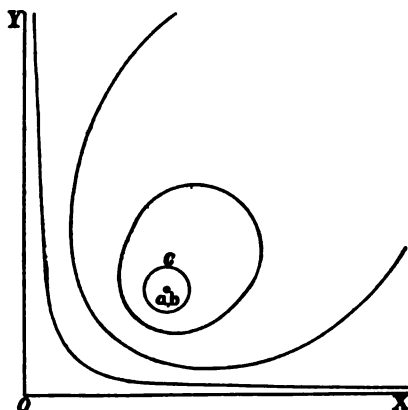


Fig. 1.

cross-section without any appreciable error if the radius r is small as compared to the distances a and b . Then the center of this circle can also be taken as the point (a, b) . From (7) we have

$$u = \frac{1}{2} \log \frac{(x^2 - y^2 - a^2 + b^2)^2 + 4(xy + ab)^2}{(x^2 - y^2 - a^2 + b^2)^2 + 4(xy - ab)^2}. \quad (8)$$

Now since we are considering only the case where r is small as compared to a and b , we can find an expression for the potential at the surface of the wire by placing $y = b$ and $x = a - r$ in (8). Then

$$u = \frac{1}{2} \log \frac{(r - 2a)^2(r^2 + 4b^2)}{r^2[(r - 2a)^2 + 4b^2]}. \quad (9)$$

Now it can be easily proved that if u in equation (8) represents the potential at the surface of the conductor, the charge on it will be one half unit. Hence the capacity per unit length between the wire and the two planes is

$$C = \frac{\frac{1}{2}}{u - u_0},$$

where u_0 is the potential of the two plane conductors. But putting

$xy = 0$ in (8) gives $u_0 = 0$, and the capacity becomes

$$C = \frac{1}{2 \log \frac{\sqrt{(r-2a)^2(r^2+4b^2)}}{r\sqrt{(r-2a)^2+4b^2}}}. \quad (10)$$

If r is small enough that it may be neglected in comparison with a in the first power, the expression reduces to

$$C = \frac{1}{2 \log \frac{2ab}{r\sqrt{a^2+b^2}}}. \quad (11)$$

This same transformation may be used for finding the capacity between two horizontal parallel wires at equal distances above the earth, the one having a positive, the other an equal negative charge, and considering the surface of the earth as an infinite conducting plane at zero potential. In this case we have for the capacity

$$C = \frac{\frac{1}{2}}{u_1 - u_2},$$

where u_1 is the potential of the wire having the positive charge and u_2 is the potential of the other wire. To find the value of u_2 , let $y = b$ and $x = -(a-r)$ and substitute in (8). Then

$$u_2 = \frac{1}{2} \log \frac{r^2[(r-2a)^2+4b^2]}{(r-2a)^2(r^2+4b^2)} \quad (12)$$

and hence $u_2 = -u_1$; which gives for the capacity

$$C = \frac{1}{\log \frac{\sqrt{(r-2a)^2(r^2+4b^2)}}{r\sqrt{(r-2a)^2+4b^2}}}, \quad (13)$$

or, neglecting r as compared to a and b , we have

$$C = \frac{1}{\log \frac{2ab}{r\sqrt{a^2+b^2}}}. \quad (14)$$

TWO WIRES PARALLEL TO THE EARTH AND ONE DIRECTLY ABOVE THE OTHER.

It was found that the equipotential lines and the lines of force about two wires parallel to the earth where one is directly above the other could be obtained by the transformation

$$w = \log \frac{(z^2 + a^2 + ad) - izd}{(z^2 + a^2 + ad) + izd}, \quad (15)$$

where a is the height of the lower wire and d is the distance between them. This is the case where the lower wire bears a positive charge and the upper wire a negative charge and the radius, r , of the wire is small compared to a and d . Then

$$u = \frac{1}{2} \log \frac{[x^2 + (y - a - d)^2][x^2 + (y + a)^2]}{[x^2 + (y + a + d)^2][x^2 + (y - a)^2]}. \quad (16)$$

The potential u_1 of the lower wire is found by putting $x = 0$ and $y = a + r$ in (16); hence

$$u_1 = \frac{1}{2} \log \frac{(d - r)^2(2a + r)^2}{(2a + d + r)^2 r^2}. \quad (17)$$

Likewise to obtain the potential of the upper wire, substitute $x = 0$ and $y = a + d + r$ in (16) which then becomes

$$u_2 = \frac{1}{2} \log \frac{r(2a + d + r)^2}{(2a + 2d + r)^2(d + r)^2}. \quad (18)$$

Whence the capacity for this system is

$$\begin{aligned} C &= \frac{\frac{1}{2}}{2 \log \frac{\sqrt{(d - r)(2a + r)(2a + 2d + r)(d + r)}}{r(2a + d + r)}}, \\ &= \frac{1}{4 \log \frac{\sqrt{(d^2 - r^2)(2a + r)(2a + 2d + r)}}{r(2a + d + r)}}. \end{aligned} \quad (19)$$

Neglecting the second power of r

$$C = \frac{1}{4 \log \frac{\sqrt{2d^2[(2a + r)(a + d) + ar]}}{r(2a + d)}}. \quad (20)$$

If we take a as infinite, we have simply the case of two parallel wires

for which the capacity is known to be

$$C = \frac{1}{4 \log \frac{l + \sqrt{l^2 - r^2}}{r}},$$

where $2l$ is the distance between them. Or neglecting r^2

$$C = \frac{1}{4 \log \frac{2l}{r}} = \frac{1}{4 \log \frac{d}{r}}, \quad (21)$$

where d is the distance between the wires. Now substituting a equal to infinity in (20) we obtain

$$C = \frac{1}{4 \log \frac{d}{r}}, \quad (22)$$

which agrees with the former result.

THREE PARALLEL WIRES ARRANGED SO AS TO BE AT THE CORNERS OF AN EQUILATERAL TRIANGLE.

In transmitting a three-phase alternating current by an overhead system, the wires are generally arranged so that they are at the corners of an equilateral triangle. We shall find the electrostatic capacity of such a system when the total charge on all the wires is zero. There will be no loss of generality if the charge on one of the wires is taken as one positive unit while that on each of the other two is taken as one half a negative unit. Again a very simple transformation can be found which will give the proper lines of force and equipotential. This transformation is

$$w = \log \frac{(z^2 - 4a^2) + 2azi}{(z^2 - 4a^2) - 4azi}, \quad (23)$$

where the distance between any two wires is $2a\sqrt{3}$. The equipotential lines will be found from the equation

$$u = \frac{1}{2} \log \frac{[(x - a\sqrt{3})^2 + (y + a)^2][(x + a\sqrt{3})^2 + (y + a)^2]}{[x^2 + (y - 2a)^2]}, \quad (24)$$

which is obtained directly from (23) in the usual way.

In the above the origin has been taken at the intersection of the medians of the triangle and the wire bearing the positive charge has its center at the point $(0, 2a)$, while the other two wires have their centers

at the points $(a\sqrt{3}, -a)$ and $(-a\sqrt{3}, -a)$ respectively. Hence to find the potential of the first wire we shall place $x = 0$ and $y = 2a - r$ in (24) as for these small values of r the equipotential lines are approximately circles. Hence we obtain

$$u_1 = \frac{1}{2} \log \frac{[3a^2 + (3a - r)^2][3a^2 + (3a + r)^2]}{r^4}, \quad (25)$$

$$= \frac{1}{2} \log \frac{(12a^2 - 6ar + r^2)^2}{r^4}. \quad (26)$$

To find the potential of the wires bearing the negative charge, let $y = -(a - \frac{1}{2}r)$ and $x = \sqrt{3}(a - \frac{1}{2}r)$ and we have

$$u_2 = \frac{1}{2} \log \frac{r^2}{(12a^2 - 6ar + r^2)}. \quad (27)$$

Hence it is seen that $u_2 = -\frac{1}{2}u_1$; and the electrostatic capacity of the three wires, which is defined to be $1/u_1 - u_2$, becomes

$$C = \frac{2}{3u_1}, \quad (28)$$

$$= \frac{1}{3 \log \left(\frac{\sqrt{12a^2 - 6ar + r^2}}{r} \right)}. \quad (29)$$

If these wires are considered parallel to the earth and the influence of the latter is considered the transformation takes the form

$$w = \log \frac{[(z - di)^2 - 3a^2][z + i(d + 3a)]}{[(z + di)^2 - 3a^2][z - i(d + 3a)]}, \quad (30)$$

where d is the height of the two lower wires above the surface of the earth. Then the expression for the potential is

$$u = \frac{1}{2} \log \frac{\{[x^2 - (y - d)^2 - 3a^2]^2 + 4x^2(y - d)^2\} \cdot [x^2 + (y + d + 3a)^2]^2}{\{[x^2 - (y + d)^2 - 3a^2]^2 + 4x^2(y + d)^2\} \cdot [x^2 + (y - d - 3a)^2]^2}. \quad (31)$$

The potential of the upper wire which bears the positive charge of one unit is found by placing $x = 0$ and $y = d + 3a - r$ in (31); which gives

$$u_1 = \frac{1}{2} \log \frac{[(3a - r)^2 + 3a^2]^2 \cdot [2d + 6a - r]^4}{r^4[(2d + 3a - r)^2 + 3a^2]^2}. \quad (32)$$

As before it can be shown that

$$u_2 = -\frac{1}{2}u_1. \quad (33)$$

Hence the capacity of this system becomes

$$C = \frac{2}{3u_1},$$

$$= \frac{1}{3 \log \frac{(2d+6a-r)}{r} \cdot \sqrt{\frac{(3a-r)^2 + 3a^2}{(2d+3a-r)^2 + 3a^2}}}. \quad (34)$$

However for all practical cases r is very small as compared to d so that we may neglect r in (34) wherever it is added to d . This gives

$$C = \frac{1}{3 \log \left[\frac{(d+3a) \sqrt{12a^2 - 6ar + r^2}}{r \sqrt{3a^2 + 3ad + d^2}} \right]}. \quad (35)$$

Placing d equal to infinite in the above equation will give the capacity of the three wires alone. This then is

$$C = \frac{1}{3 \log \left(\frac{\sqrt{12a^2 - 6ar + r^2}}{r} \right)}, \quad (36)$$

which is the same value found in equation (29).

It is also easily seen that the value of the capacity found in (35) is larger than that in (29), as was to be expected.

IMAGES IN A CYLINDER.

Let us consider the transformation

$$w = \log \frac{R^2 - az}{R(z - a)}. \quad (37)$$

This gives for u

$$u = \log \frac{\sqrt{(R^2 - ax)^2 + a^2y^2}}{R\sqrt{(x - a)^2 + y^2}}. \quad (38)$$

For $u = 0$, we have

$$(R^2 - ax)^2 + a^2y^2 = R^2[(x - a)^2 + y^2],$$

whence

$$x^2 + y^2 = R^2. \quad (39)$$

That is, the zero potential surface is a cylinder and the cross-section in the xy -plane is a circle of radius R . Again from equation (38), we obtain

$$e^{2u} = \frac{(R^2 - ax)^2 + a^2y^2}{R^2[(x - a)^2 + y^2]}$$

and

$$x^2 + y^2 - 2x \cdot aR^2 \frac{e^{2u} - 1}{R^2 e^{2u} - a^2} + R^2 \frac{a^2 e^{2u} - R^2}{R^2 e^{2u} - a^2} = 0,$$

or

$$\left(x - aR^2 \frac{e^{2u} - 1}{R^2 e^{2u} - a^2} \right)^2 + y^2 = \left[\frac{e^u R(a^2 - R^2)}{R^2 e^{2u} - a^2} \right]^2. \quad (40)$$

Hence the equipotential lines are all circles with their centers on the x -axis and at a distance from the origin given by

$$d = aR^2 \frac{e^{2u} - 1}{R^2 e^{2u} - a^2}. \quad (41)$$

The radii are given by

$$r = \pm \frac{e^u R(a^2 - R^2)}{R^2 e^{2u} - a^2}, \quad (42)$$

in which the positive sign is to be used for values of u greater than $u = \log(a/r)$ and the negative sign for values less than that. The

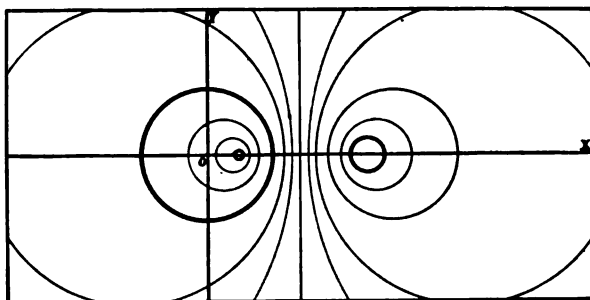


Fig. 2.

reason for this is quite obvious from Fig. 2. The radius becomes infinite for $u = \log(a/r)$ and the equipotential curve becomes the straight line

$$x = \frac{a^2 + R^2}{2a}.$$

Now let r_1 be the radius of a small wire bearing a charge of one half unit per unit length placed parallel to a large, earthed, cylindrical conductor of radius R and let d be the distance between the centers of the two. Then we can replace the cylinder by another wire bearing a negative charge of one half unit without any change in the electrostatic field. To find the radius r_2 and the position of this latter wire replace u by $-u$ in equations (41) and (42). It will be seen that r_2 is less than r_1 as might

be expected. For

$$r_1 = \frac{e^u R(a^2 - R^2)}{R^2 e^{2u} - a^2}, \quad r_2 = \frac{e^u R(a^2 - R^2)}{a^2 e^{2u} - R^2}. \quad (43)$$

Also the positions of the centers of the wires will be given by

$$d_1 = aR^2 \frac{e^{2u} - 1}{R^2 e^{2u} - a^2}, \quad d_2 = aR^2 \frac{e^{2u} - 1}{a^2 e^{2u} - R^2}. \quad (44)$$

It is to be understood that the above reasoning only holds for the case where very long and large cylinders are considered. It can readily be shown that the method of images does not apply rigorously to the case of two long small parallel wires at any considerable distance. That is, as suggested earlier in this paper, the electric force in the direction of the z -axis must be so small that it may be neglected. However in this as in the general theory of the logarithmic potential, the very long wires must be considered as finite in length when applying the test of zero potential at an infinite distance.

Now equation (38) may be used for finding the capacity between a wire bearing a charge and an earthed wire near it. First consider the case where the small wire is external to the earthed cylinder. The first of equations (43) gives

$$r_1 = \frac{e^u R(a^2 - R^2)}{R^2 e^{2u} - a^2},$$

from which we find

$$a^2 = \frac{R^2(e^u R + r_1 e^{2u})}{e^u R + r_1}. \quad (46)$$

From the first of equations (45)

$$d_1 = aR^2 \frac{e^{2u} - 1}{R^2 e^{2u} - a^2},$$

whence

$$d_1^2 = \frac{(R + r_1 e^u)(e^u R + r_1)}{e^u}.$$

Solving for e^u ,

$$e^u = \frac{(d_1^2 - r_1^2 - R^2) + \sqrt{(d_1^2 - r_1^2 - R^2)^2 - 4R^2 r_1^2}}{2Rr_1}, \quad (47)$$

and

$$u = \log \frac{(d_1^2 - r_1^2 - R^2) + \sqrt{(d_1^2 - r_1^2 - R^2)^2 - 4R^2 r_1^2}}{2Rr_1}. \quad (48)$$

Now it is easily proved that the charge on the wire is one half a unit, so that the capacity of this system is

$$C_1 = \frac{I}{2 \log \frac{(d_1^2 - r_1^2 - R^2) + \sqrt{(d_1^2 - r_1^2 - R^2)^2 - 4R^2r_1^2}}{2Rr_1}}. \quad (49)$$

Likewise the capacity between the internal wire and the cylinder may be found from the second of the equations (44) and (45) respectively. This gives

$$C_2 = \frac{I}{2 \log \frac{(R^2 + r_2^2 - d_2^2) + \sqrt{(R^2 + r_2^2 - d_2^2)^2 - 4r_2^2R^2}}{2r_2R}}. \quad (50)$$

To prove that the above expressions have the proper form, place $d = 1 + R$ and let R become infinite; that is, let the earthed cylinder become an infinite earthed plane. This gives

$$C = \frac{I}{2 \log \frac{l + \sqrt{l^2 - r^2}}{r}},$$

which is the identical expression previously obtained for a wire parallel to an infinite earthed plane.

An expression for the surface distribution over the earthed cylinder is very easily obtained. From equation (37) we obtain

$$\frac{dw}{dz} = \frac{R^2 - a^2}{az^2 - z(R^2 + a^2) + aR^2}.$$

Taking the absolute value and simplifying

$$\left| \frac{dw}{dz} \right| = \frac{R^2 - a^2}{\sqrt{[a(x^2 - y^2 + R^2) - x(R^2 + a^2)]^2 + [y(2ax - R^2 - a^2)]^2}}.$$

Transforming to polar coördinates

$$\left| \frac{dw}{dz} \right| = \frac{R^2 - a^2}{R(R^2 + a^2 - 2aR \cos \varphi)}.$$

Hence

$$\sigma = \frac{I}{4\pi} \frac{R^2 - a^2}{R(R^2 + a^2 - 2aR \cos \varphi)} \quad (51)$$

and a can be calculated for any particular system from equation (46). It is seen that the maximum value of σ is for $\varphi = 0$ and the minimum value for $\varphi = \pi$.

TWO CORE CABLE.

We shall next consider the transformation

$$w = \log \frac{(z+a)(z-b)}{(z-a)(z+b)}.$$

This gives

$$u = \frac{1}{2} \log \frac{[(x+a)^2 + y^2][(x-b)^2 + y^2]}{[(x-a)^2 + y^2][(x+b)^2 + y^2]},$$

from which the zero potential curves are found to be

$$x = 0$$

and

$$x^2 + y^2 = ab.$$

The latter suggests the possibility of using the transformation for the cases where wires are encased in an earthed sheath of circular cross section, and where the total charge on the cable at any instant is zero.

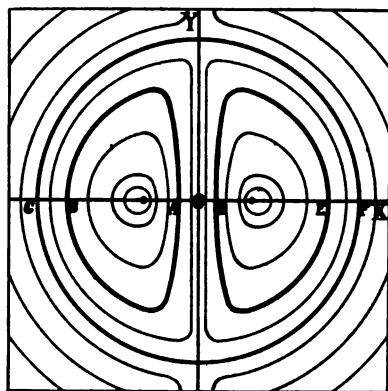


Fig. 3.

The following results are for air as the dielectric, but of course the results can be altered to fit the case for any dielectric.

If the radius of this sheath is R , we have

$$ab = R^2.$$

Replacing b in the original equation by (R^2/a) gives

$$w = \log \frac{(z+a)(az - R^2)}{(z-a)(az + R^2)}, \quad (52)$$

and for the potential

$$u = \frac{1}{2} \log \frac{[(x+a)^2 + y^2][(ax - R^2)^2 + a^2y^2]}{[(x-a)^2 + y^2][(ax + R^2)^2 + a^2y^2]}. \quad (53)$$

The equation of the equipotential lines becomes

$$e^{2u}[(x-a)^2 + y^2][(ax + R^2)^2 + a^2y^2] = [(x+a)^2 + y^2][(ax - R^2)^2 + a^2y^2].$$

These curves are shown in Fig. 3. In many cables,¹ used especially in Europe, a bundle of wires is so arranged that the contour exactly coincides with the heavy curves AB and DE shown in Fig. 3.

If $a = R/3$, we shall have $AD = EF$. This is the most usual condition. Substituting $R/3$ for a in (53), we obtain

$$u = \frac{1}{2} \log \frac{\left[\left(x + \frac{R}{3}\right)^2 + y^2\right] \left[\left(\frac{Rx}{3} - R^2\right)^2 + \frac{R^2y^2}{9}\right]}{\left[\left(x - \frac{R}{3}\right)^2 + y^2\right] \left[\left(\frac{Rx}{3} + R^2\right)^2 + \frac{R^2y^2}{9}\right]}. \quad (54)$$

Now let $AD = EF = 2d$; then for $y = 0$, $x = d$ we obtain the potential of the bundle of wires DE , which is

$$\begin{aligned} u_1 &= \log \frac{(3d + R)(d - 3R)}{(3d - R)(d + 3R)}, \\ &= \log \frac{3R^2 + 8Rd - 3d^2}{3R^2 - 8Rd - 3d^2}. \end{aligned} \quad (55)$$

Likewise the potential of the bundle of wires AB will be obtained by placing $y = 0$ and $x = -d$. Hence

$$u_2 = \log \frac{(3d - R)(d + 3R)}{(3d + R)(d - 3R)}.$$

Hence $u_2 = -u_1$ and the capacity per unit length between the two bundles of wires, one positively the other negatively charged, surrounded by an earthed cylinder of circular cross-section is

$$\begin{aligned} C &= \frac{\frac{1}{2}}{2u_1}, \\ &= \frac{1}{4 \log \left[\frac{3(R^2 - d^2) + 8Rd}{3(R^2 - d^2) - 8Rd} \right]}. \end{aligned} \quad (56)$$

Another case of interest is that in which two small wires are encased in the circular sheath. As is seen in Fig. 3, the smaller equipotential lines approximate to circles. Making this approximation, we shall

¹ See Russell's *Alternating Currents*, Vol. I, Chapters IV. and V.

obtain the potential of the wire bearing the positive charge by placing $y = 0$ and $x = a - 2r/3$ in (53), which gives the case for $a = R/3$. Hence

$$u = \log \frac{(2r - 6a)(3a^2 - 2ar - 3R^2)}{2r(3a^2 - 2ar + 3R^2)}. \quad (57)$$

Then placing $a = R/3$, we have

$$u = \log \frac{(R - r)(4R - r)}{r(5R - r)}.$$

And the capacity between the two wires within the earthed sheath is

$$C = \frac{1}{4 \log \frac{(R - r)(4R - r)}{r(5R - r)}}. \quad (58)$$

It is to be understood that this latter form is an approximation that can be used only for small wires so placed that the distance from the surface of one wire to the surface of the other is equal to the distance from the inner surface of the metal sheath to the surface of the nearest wire. However the method can be adapted to other conditions by taking the proper relations between a , r , and R .

THREE CORE CABLES.

A transformation was found which would give equipotential lines which very nearly coincide with those in the clove-leaf type of three-core cable. A diagram of this cable is shown in Russell's *Alternating Currents*, Vol. I. This transformation is

$$w = \log \frac{[(z^2 - R^2) + Rzi] \cdot (z - 4Ri)^2}{[(z^2 - 16R^2) + 4Rzi](z - Ri)^2}. \quad (59)$$

where the distance between the surfaces of the bundles of wires is equal to the distance from the inner surface of the enclosing metal sheath to the surface of one bundle of wires. The charge on one of the conductors is one positive unit, while each of the other two has a negative charge of one half unit. Then the equation of an equipotential line becomes

$$u = \frac{1}{2} \log \frac{[(x^2 - y^2 - R^2 - Ry)^2 + x^2(2y + R)^2][x^2 + (y - 4R^2)^2]}{[(x^2 - y^2 - 16R^2 - 4Ry)^2 + x^2(2y + 4R)^2][x^2 + (y - R)^2]}. \quad (60)$$

Now if d is the distance from the center of the circular cylinder to the

outer surfaces of the bundles of wires, an expression for the potential of the wire bearing the positive charge will be obtained by substituting $x = 0$ and $y = d$ in (60). This gives

$$\begin{aligned} u_1 &= \log \frac{(R^2 + d^2 + Rd)(4R - d)^2}{(16R^2 + d^2 + 4Rd)(R - d)^2}, \\ &= 2 \log \left[\frac{(4R - d)\sqrt{(R + d)^2 - Rd}}{(R - d)\sqrt{(4R + d)^2 - 4Rd}} \right]. \end{aligned} \quad (61)$$

Now it can be proved that the potential of the conductors bearing the negative charge is

$$u_2 = -\frac{1}{2}u_1.$$

Hence the capacity of the system is

$$C = \frac{1}{3 \log \left(\frac{4R - d}{R - d} \cdot \sqrt{\frac{(R + d)^2 - Rd}{(4R + d)^2 - 4Rd}} \right)}. \quad (62)$$

If the conductors are small wires of circular cross-section, and radius r , a close approximation to the value of the capacity will be obtained by substituting $d = \frac{1}{2}R + r$ in (62). For this condition then

$$C = \frac{1}{3 \log \left(\frac{7R - 2r}{2R - 4r} \cdot \sqrt{\frac{7R^2 + 8Rr + 4r^2}{18R^2 + 5Rr + r^2}} \right)}. \quad (63)$$

It may seem at first that too many approximations have been made in this paper, but a closer examination will show that the results obtained by using the formulæ derived will, for nearly all practical cases, be more accurate than the measurements from which the calculations are made. In the cases of the cables if the wires have the shape of the equipotential lines, the results will be exact.

In conclusion I wish to thank Dr. J. Kunz and Professor E. J. Townend for their many helpful suggestions during the investigation of the above problems.

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August, 1912.

SOME REMARKABLE CASES OF RESONANCE.¹

BY C. V. RAMAN.

THE general principle of resonance is that a periodic force acting on an oscillatory system may set up and maintain a large amplitude of vibration when the periods of the force and of the system are approximately equal, even though in other cases the amplitude might be so small as to be negligible. In the present paper I propose to discuss some remarkable cases which form apparent exceptions to this law of equality of periods, that is, in which we have marked resonance when the periods of the impressed force and of the system do *not* stand to each other in a relation of approximate equality.

The first of these is the well-known case of double frequency, the theory of which was first discussed by Lord Rayleigh.² In a note published in the *PHYSICAL REVIEW* for March, 1911, I promised a fuller discussion of the modification in this theory necessary to fit the results with those actually observed in experiment. I now proceed to fulfil this promise and the delay that has occurred in doing so is I feel a matter for regret. Lord Rayleigh starts with the following as his equation of motion:

$$\ddot{U} + k\dot{U} + (n^2 - 2\alpha \sin 2pt)U = 0, \quad (1)$$

and assuming that U , the displacement at any instant during steady motion, can be represented by an expression of the form

$$A_1 \sin pt + B_1 \cos pt + A_3 \sin 3pt + B_3 \cos 3pt + A_5 \sin 5pt, \quad (2)$$

proceeds to find the conditions that must be satisfied for the assumed steady motion to be possible. This he does by substituting (2) for U in the left-hand side of equation (1) and equating to zero the coefficients of $\sin pt$, $\cos pt$, etc. The conditions for the possibility of steady motion thus obtained are

$$\frac{B_1}{A_1} = \frac{\sqrt{(\alpha - kp)}}{\sqrt{(\alpha + kp)}} = \tan e, \quad (3)$$

$$(n^2 - p^2)^2 = \alpha^2 - k^2 p^2. \quad (4)$$

By a trigonometrical transformation equation (3) can be written in the

¹ Preliminary notes on this subject appeared in *Nature*, December 9, 1909, and February 10, 1910, and in the *PHYSICAL REVIEW*, March, 1911.

² *Phil. Mag.*, April, 1883, and August, 1887, and *Theory of Sound*, Art. 68 (b).

form

$$kp = \alpha \cos 2e. \quad (5)$$

It appears from these equations that the phase of the oscillation maintained, *i. e.*, e is independent of the amplitude, and that the latter quantity is indeterminate.

I attempted to verify the phase-relation given by equations (4) and (5) experimentally in the following way: The oscillating system used was a stretched string and this was maintained in motion by periodically varying its tension in the manner of Melde's experiment. Since the periodic change of double frequency in the tension of the string is imposed by the tuning-fork, the motion of the prong corresponds to the term $-2\alpha \sin 2pt$ in equation (1) and the transverse vibration of the string to the expression $(A_1^2 + B_1^2)^{1/2} \sin(pt + e)$ if the small terms in A_1 , B_1 , etc., are neglected. The experimental problem therefore reduces itself to the determination of the phase-relation between the motion of the string and the vibration of the prong of the exciting tuning-fork. This can be attacked by two distinct methods. The first is

Mechanical Composition of the two motions. This is automatically effected and needs no special experimental device. For, the motion of the prong is longitudinal to the string and any point on the string near the end attached to the prong or near any other intermediate node of the oscillation has *two* rectangular motions: the first longitudinal to the string and having the same frequency as the vibration of the fork; and the second which is the general transverse oscillation of the string. The resulting motion is in a Lissajous figure and this is readily observed by attaching a fragment of a silvered bead to a point on the string near the fork.

Optical Composition of the two motions furnishes a second method and this is undoubtedly the more elegant of the two. It can be effected in the following way: A small mirror is attached normally to the extremity of the prong of the fork. The plane of the oscillation of this mirror is perpendicular to that of the vibration of the string, and a point on the latter is brightly illuminated by a transverse sheet of light from a lantern or from a cylindrical lens. When the string is in oscillation the illuminated point appears drawn out into a straight line, and this is viewed by reflection first at a fixed mirror and then at the mirror attached to the vibrating prong. The illuminated point is then seen to describe a Lissajous figure which is compounded of the motions of the string and the tuning-fork.

Observing by either of the methods described above, the relation

between the phases of the transverse oscillation of the string and of the motion of the prong of the tuning-fork can be closely studied, and some remarkable phenomena are noticed in this way. The principal point observed is that the phase relation is not independent of the amplitude maintained. This is best shown by using a bowed fork and starting with a large amplitude of motion and then allowing the motion to die away. The initial curve of motion and the changes in it as the motion dies away both depend on the tension of the string. When this is in excess of that required for the most vigorous maintenance, the curve is a parabolic arc convex to the fork and remains as such when the motion dies away. With a smaller tension adjusted so that the free period of the string is exactly double that of the fork, the initial curve with a large amplitude of motion is still approximately a parabolic arc convex to the fork. The damping of the motion is now more rapid and the curve reduces to an 8-shaped figure when the amplitude is very small. For the most vigorous maintenance a still smaller tension is necessary and the initial curve with a large motion is still convex to the fork, but it will now be noticed that when the amplitude falls to a very small quantity the curve passes through the 8-figure stage and tends to become *concave* to the fork. The most remarkable changes are observed when the tension is smaller still. The damping here is very large and a steady motion is only possible when the amplitude exceeds a certain minimum value. At this stage the string very rapidly comes to rest and in the final stage the curve of motion becomes a parabolic arc *concave* to the fork with a very small amplitude. For a satisfactory explanation of these phenomena it is necessary to start with a modified equation of motion which takes into account the variations of tension which exist in *free* oscillations of sensible amplitude and are proportional to the square of the motion. The equation of motion thus completed is

$$\ddot{U} + k\dot{U} + (n^2 - 2\alpha \sin 2pt + \beta U^2) \cdot U = 0. \quad (6)$$

Substituting expression (2) for U in the left-hand side of the above given equation and putting the coefficients of $\sin pt$, $\cos pt$, etc., equal to zero, the conditions for the possibility of steady motion reduce to the form

$$kp = \alpha \cos 2e \quad (7)$$

and

$$(n^2 - p^2 + F)^2 = \alpha^2 - k^2 p^2, \quad (8)$$

where

$$F = \frac{3\beta}{4} (A_1^2 + B_1^2).$$

Equation (8) determines the amplitude of the motion and (7) its phase. It is evident at once that with given values for α and kp the amplitude of the motion is greatest when n is smallest and that there can be no maintenance if $\alpha < kp$. We therefore get the apparently paradoxical conclusion (which is amply verified by experiment) that the maintenance is *not* the most vigorous when the free period (for small amplitudes) of the string is double that of the fork. Another interesting inference which is confirmed by experiment is that when $n < p$ and $(n^2 - p^2)^2 > (\alpha^2 - k^2 p^2)$ maintenance is impossible unless F , *i. e.*, also the amplitude, has a definite minimum value.

Equation (7) shows that as α is increased e , the phase of the oscillation, alters continuously. The influence of F , *i. e.*, of the amplitude of the motion on e its phase can readily be traced from equation (8). When $n > p$, for a larger value of F we must have a large value of α and $\cos 2e$ tends more and more to assume a zero value. Writing equation (7) in the form

$$\tan e = \frac{\alpha - kp}{n^2 - p^2 + F} \quad (9)$$

it is evident that when $n > p$ and F is large, e is positive and approaches to the value $\pi/4$. This agrees with the experimental result. When $n < p$, e may be positive or negative according as F is greater or less than $(p^2 - n^2)$ and the alteration of the phase of the motion with the amplitude is most conspicuous, and this is in agreement with observation. In the extreme lower limit e tends to the value $-\pi/4$, and the curve of motion is a parabolic arc *concave* to the fork. In other words when the prong is at its extreme outward swing, the string is also at its extreme outward swing, a seemingly paradoxical result not in accordance with the ordinarily received ideas of the experiment. Equation (1) may be written in the form

$$\ddot{U} + k\dot{U} + n^2 U = 2\alpha U \sin 2pt. \quad (10)$$

The right-hand side of this equation may be regarded as the *impressed* part of the restoring force acting on the system, and this is a very useful way of regarding the matter. Putting $U = P \sin (pt + e)$ to a first approximation, we can find the conditions that must exist for steady motion *directly* by equating the work done by the force represented by the right-hand side term of equation (10) to the energy dissipated in an equal time by the friction term on the left. The relation thus obtained is

$$kp = \alpha \cos 2e,$$

which is identical with equation (7) obtained from the complete analysis. It is observed that the right-hand side of equation (10) is

$$2\alpha P \sin 2pt \sin (pt + e)$$

and this may be written as

$$\alpha P (\cos pt - e - \cos 3pt + e).$$

The second term within the brackets is ineffective so far as the maintenance of the motion $P \sin (pt + e)$ is concerned. We may therefore leave it out and write equation (10) in the form

$$\ddot{U} + k\dot{U} + n^2U = \alpha P \cos (pt - e). \quad (11)$$

Written in this way it is evident that a large motion must ensue if $p = n$ and that we have here merely an example of the general principle of resonance.

PART II.

I now proceed to consider some other exceedingly interesting cases of resonance under the action of forces similar in character to that in the case of double frequency considered above, but having other frequency relations to the system influenced. My experiments show that resonance may occur in the following cases of the kind:

- (1) When the period of the force is $\frac{1}{2}$ that of the system.
- (2) When the period of the force is $\frac{2}{3}$ times that of system.
- (3) When the period of the force is $\frac{3}{4}$ times that of the system.
- (4) When the period of the force is $\frac{4}{5}$ times that of the system.
- (5) When the period of the force is $\frac{5}{6}$ times that of the system.
- (6) When the period of the force is $\frac{6}{7}$ times that of the system.

And so on.

The most remarkable instances of these cases of resonance are furnished by a stretched string under the action of a periodically varying tension. To observe them, all that is required is that the tension of the string should be gradually increased till its free period in the fundamental mode stands in the desired relation to the period of the tuning-fork which imposes the variable tension. It will then be found that a vigorous oscillation is maintained. Figs. 1, 2, 3, 4 and 5 are photographs of stretched strings maintained in the first five types of motion respectively under the action of an electrically maintained tuning fork varying the tension periodically.

The actual frequency and the phase of the maintained motion in each of these cases can be determined by observation of the corresponding Lissajous figures, using the mechanical or optical method of composition described above for the first of these cases. The detailed results I must reserve for a future paper. One good way of studying these types of motion is to illuminate one point on the vibrating string by a transverse sheet of light from a lantern or from a cylindrical lens and to observe the line of light so produced in a revolving mirror. But the best method of all for recording the motion photographically is that by which I obtained Figs. 6, 7, 8, 8 (*a*), 9 and 10 published herewith and which I now proceed to describe.

Figs. 6, 7, 8, 9 and 10 refer respectively to the first five types of motion as shown in Figs. 1, 2, 3, 4 and 5. It will be observed that each of them shows two curves. The white curve in the black ground is a record of the motion of the tuning-fork, and the other curve which is black on a white ground is a record of the motion of a point on the string maintained in vibration. These records were obtained on a moving photographic plate in the following manner. One source of light was a horizontal slit, and the other was a vertical slit placed behind the oscillating string. Both were illuminated by sunlight and had collimating lenses in front of them. The light from the former fell upon a small mirror attached to the prong of the vibrating fork and after reflection fell upon the lens (having an aperture of 4 cm. diameter) of a roughly constructed camera. The light from the vertical slit behind the vibrating string was also reflected into the camera by a fixed mirror. In the focal plane of the camera was placed a brass plate with a vertical slit cut in it. The images of the horizontal and vertical slits fell, one immediately above the other, on the slit in the plate. Only a very small length of the former, *i. e.*, practically only a point of light was allowed to fall upon the photographic plate. The dark slide which held this was moved uniformly by hand in horizontal grooves behind the slit in the focal plane of the camera, while the fork and the string were in oscillation.

Figs. 1 and 6 represent the well-known case in which the string makes one oscillation for every two oscillations of the fork. This is evident in the photograph.

Figs. 2 and 7 represent the next type in which the variable tension maintains an oscillation of the same frequency as its own. It will be noticed that the curvature of one of the extreme positions of the string is somewhat greater than that of the other and that the mid-point of the oscillation is somewhat displaced to one side of the middle-point of the vertical slit at which the string was set when at rest. The inference

from this fact of observation is that the transverse motion of each point on the string is represented by an expression of the form

$$P \sin (2pt + e) + Q, \quad (12)$$

where Q is a constant. A motion of this type is only possible under a variable spring. For, the restoring forces acting on an element of the string at the two unequally curved extremes of its swing cannot themselves be equal and opposite (the condition necessary for a simple harmonic oscillation) unless the tensions of the string at the two extreme positions are unequal. In fact the second constant term Q in the motion is introduced under the action of the variable spring, and its importance will become evident as we proceed.

Fig. 3 and 8 and 8 (*a*) represent the third type of motion in which the string makes *three* swings for every *two* swings of the fork. But it is evident from Figs. 8 and 8 (*a*) that the successive swings on opposite sides are not all equal in amplitude and the influence of this is also perceptible in Fig. 3, having given rise to the appearance of two extra strings, which represent really the turning points of the motion. The vibration curve shown in Figs. 8 and 8 (*a*) can be represented by an expression of the form

$$P \sin (3pt + e) + Q \sin (pt + e'). \quad (13)$$

The alternate increase and decrease of the amplitude of the motion of the string is evidently due to the action of the varying tension and the term $Q \sin (pt + e')$ in the motion which superposed on the first reproduces this waning and waxing effect, plays a very important part in the maintenance of the motion, as we shall see later on. Figs. 4 and 9 represent the fourth type of motion in which the string makes *four* swings for every *two* swings of the fork. As before the waning and waxing of the motion under the action of the variable spring is evident in the photographs and the observed motion of the points on the string is of the type

$$P \sin (4pt + e) + Q \sin (2pt + e'). \quad (14)$$

Here as before we shall see that the second term which is introduced under the action of the variable spring plays a very important part in the maintenance of the motion.

Figs. 5 and 10 represent the fifth type of motion in which the string makes five swings for every two swings of the fork. The periodic increase and decrease in the amplitude of the motion is also evident. The vibration-curve may be represented by

$$P \sin (5pt + e) + Q \sin (3pt + e'), \quad (15)$$

the second term being due to the action of the variable tension.

In the general case therefore we may assume the maintained motion to be of the form

$$P \sin (\gamma pt + e) + Q \sin (\overline{\gamma - 2pt} + e'). \quad (16)$$

The equation of motion under a variable spring may be written as

$$\ddot{U} + k\dot{U} + n^2U = 2\alpha U \sin 2pt.$$

(see equation (10) above).

If now we substitute (16) for U , the right-hand side of equation (10) gives us what we may regard as the impressed part of the restoring force at any instant. It may be written as

$$\begin{aligned} &\alpha P [\cos (\overline{\gamma - 2pt} + e) - \cos (\overline{\gamma + 2pt} + e)] \\ &+ \alpha Q [\cos (\overline{\gamma - 4pt} + e') - \cos (\gamma pt + e')]. \end{aligned} \quad (17)$$

The work done by this force in a period of time t embracing any number of complete cycles of the variable spring is found on integration to be equal to $PQ\alpha pt \cos (\pi + e - e')$ if $\gamma > 2$ or to $2PQ\alpha pt \sin e \sin (e' - \pi)$ if $\gamma = 2$. The surplus of energy thus made available may be sufficient to counteract the loss by dissipation in the same time, *i. e.*, to maintain the motion.

It is not difficult to make out from equations (10) and (17) that these apparently anomalous cases in reality form illustrations of the general principle of equality of periods required for resonance. For, we get a large motion when $n = \gamma p$ in the general case, and the reason for this is evident at once if we neglect the first three terms in (17) as ineffective and write equation (10) as under

$$\ddot{U} + k\dot{U} + n^2U = -\alpha Q \cos (\gamma pt + e'). \quad (18)$$

We started on the assumption that

$$U = P \sin (\gamma pt + e) + Q \sin (\overline{\gamma - 2pt} + e').$$

Equation (18) shows that if we had neglected the second term (coefficient Q) we should have been unable to account for the resonance effect observed. Probably for a more complete discussion it would be necessary to take three terms thus:

$$U = P \sin (\gamma pt + e) + Q \sin (\overline{\gamma - 2pt} + e') + R \sin (\overline{\gamma + 2pt} + e_1).$$

Experiment shows however that the third term (with coefficient R) is relatively unimportant and the treatment given above may therefore be taken as a fairly close first approximation. It will be noticed from Figs. 7, 8, 8 (a), 9 and 10 that the epochs of maximum amplitude in each case pretty closely correspond to those of minimum tension and vice versa. This is exactly what is to be expected on general considerations.

Some very curious phenomena are observed when the vibration of the string in each of the cases described above is observed through a stroboscopic disk. These and other matters I hope to detail in a future paper.

PART III.

In part II. the equations of motion discussed are throughout those of a body having one degree of freedom. This was sufficient for the purpose of elucidating the leading features of each type of motion considered. But it must not be overlooked that the systems dealt with, *i. e.*, stretched strings, have more than one normal mode of motion and this fact leads to certain exceedingly interesting complications. The phenomena observed under this heading fall into two distinct classes which I shall discuss separately.

The first class of phenomena I have designated "transitional types of oscillation." Their existence may be explained somewhat as follows: Take the case of a system maintained in one of its natural modes of vibration by periodic forces of double frequency. It is evident that the actual period of vibration would be *exactly* double the period of the acting force but the *free* period of vibration in the particular mode may differ slightly from the *forced* period of vibration. The range and extent of the permissible difference between the two is a function of the magnitude of the periodic force acting on the system. Assume now that the system has another natural mode of vibration whose frequency for free oscillations is not very far removed from that of the first and that the magnitude and frequency of the periodic force acting on the system is such that the ranges of the two natural modes of vibration for maintenance by forces of double frequency partly overlap and the force actually at work falls within the overlapping part. It is evident that in such a case the system would vibrate with a frequency equal to exactly half that of the acting force, but the *mode* of vibration would not be either of the natural modes but something intermediate between the two. These "transitional" types or modes of motion possess special experimental interest in the case of stretched strings as they can be readily observed and studied. It is not at all difficult for instance to maintain a "transition" mode of oscillation intermediate between the ordinary modes with three and four ventral segments respectively, by suitably adjusting the tension and varying it periodically by the aid of a tuning-fork. The frequency of the motion would everywhere be exactly half that of the fork and the motion at each point strictly "simple harmonic," but there would be no "nodes" or points of rest visible. Such a type of motion presents a very remarkable appearance when examined under

intermittent illumination of periodicity slightly different from that of the tuning-fork. The two intersecting positions of the string seen undergo a periodic cycle of changes enclosing alternately three and four ventral segments.

The phenomena observed in the experiment described above can be explained on the supposition that the displacement at each point can be represented by the equation

$$y = a \sin \frac{3\pi x}{l} \sin (pt + e) + b \sin \frac{4\pi x}{l} \sin (pt + e'). \quad (19)$$

Equation (19) suggests that the phase of the motion is not the same at all points of the string. In fact working by the optical and mechanical methods described in the first part of this paper I observed very remarkable variations of phase over the length of the string. It appeared that in some cases e and e' differed by as much as $\pi/2$.

Of course we should get "transitional types" of oscillation with the vibrations of higher frequencies maintained by periodic forces which were discussed in part II. of this paper, but they are not so marked as in the case of double frequency since the frequency-ranges become smaller as we go up the series.

The second class of phenomena observed cannot be fully discussed within the limits of the present paper and I shall have to content myself with briefly indicating their nature. In part II. of this paper I showed that a variable tension or "spring" may under suitable circumstances maintain an oscillation of a frequency standing in any one of a series of ratios to its own frequency. If the system which is subject to the variable "spring" or tension has itself a series of natural modes or frequencies, it would evidently be possible for two or more modes of vibration to be set up simultaneously with the respective frequencies and we would find a "simple harmonic" variation of tension maintaining a *compound* vibration. The special interest of this in the case of stretched strings consists of the fact that the natural frequencies of the system themselves form a harmonic series, and we may also have oscillations set up independently by one and the same force in rectangular planes and the compound character of the motion would be rendered visible by the curved paths of points on the string. These curves would in fact be identical with or analogous to the respective Lissajous figures and I hope with a future paper to publish several photographs which I have taken of compound vibrations maintained in this manner by a single tuning-fork. Two of these will be found published with my note in Nature, Feb. 10, 1910.

FIG. 1.

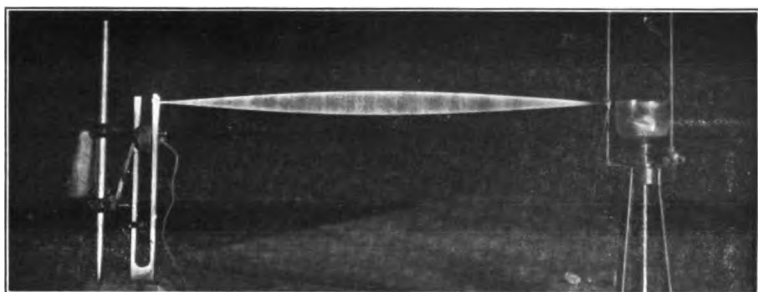


FIG. 2.

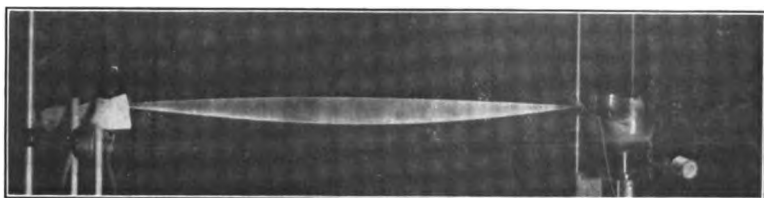


FIG. 3.

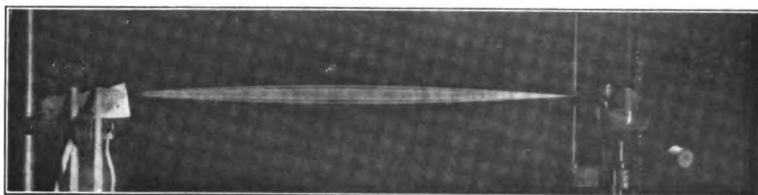


FIG. 4.

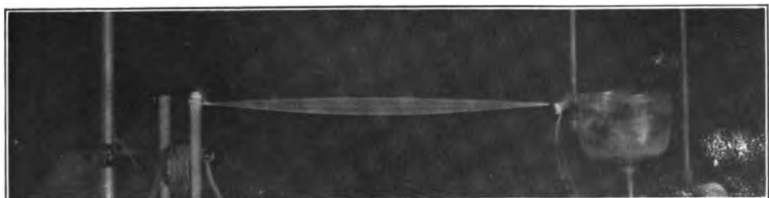
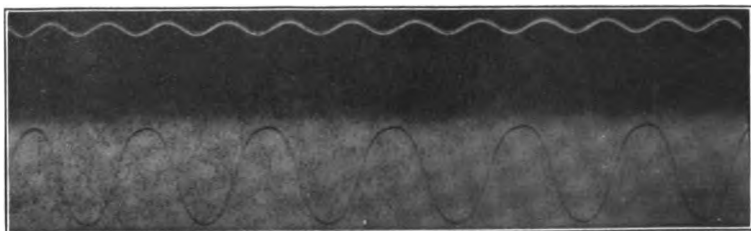


FIG. 5.



FIG. 6.



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FIG. 7.

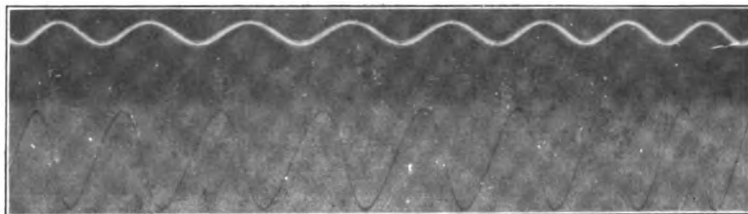


FIG. 8.

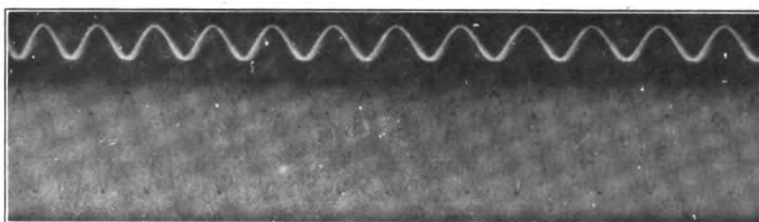


FIG. 8a.

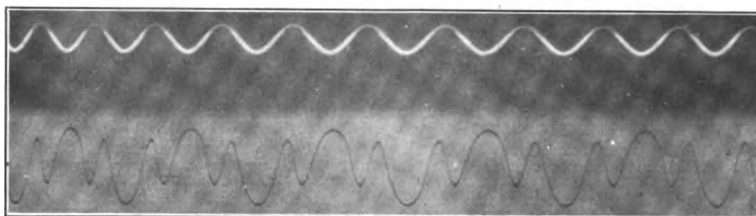


FIG. 9.

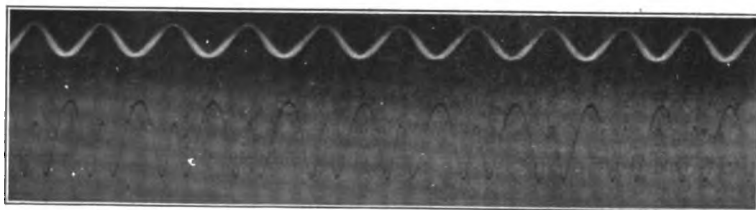
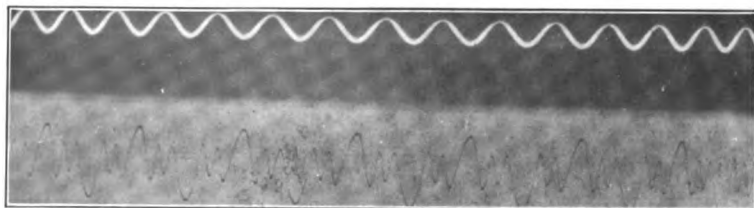


FIG. 10.



C. V. RAMAN.

THE POTENTIAL AND ELECTROSTATIC FORCE IN THE FIELD OF TWO METAL SPHERICAL ELECTRODES.

BY GEO. R. DEAN.

I. THE POTENTIAL AT ANY POINT.

TWO conducting spheres are charged to potentials V_1 and V_2 . The ratio of the distance of any point from the two inverse points of the spheres being denoted by e^η and the angle between them by ξ , the potential at the point (η, ξ) is

$$V_1 \{ \sqrt{2 (\cosh \eta - \cos \xi)} \} \sum_0^\infty \frac{\sinh (m + \frac{1}{2})(\beta + \eta)}{\sinh (m + \frac{1}{2})(\alpha + \beta)} P_m (\cos \xi) e^{-(m+1/2)\alpha} \\ + V_2 \{ \sqrt{2 (\cosh \eta - \cos \xi)} \} \sum_0^\infty \frac{\sinh (m + \frac{1}{2})(\alpha - \eta)}{\sinh (m + \frac{1}{2})(\alpha + \beta)} P_m (\cos \xi) e^{-(m+1/2)\beta}, \quad (1)$$

where $P_m (\cos \xi)$ is a zonal harmonic, and $\eta = \alpha$ on the surface of the sphere whose potential is V_1 and $\eta = -\beta$ on the surface of the sphere whose potential is V_2 .

Before giving the details of the derivation of (1) it will make the matter more interesting to the reader to give an outline of the process.

The function (1) is a solution of Laplace's equation, which in the case of symmetry with respect to an axis, as in the case of the spheres, is

$$\frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{z} \frac{\partial V}{\partial z} = 0, \quad (2)$$

r being measured along the axis, and z normal to it.

The first step is to transform (2) to new independent variables defined by the equations

$$n = \frac{1}{2} \log \left[\frac{z^2 + (c + r)^2}{z^2 + (c - r)^2} \right], \quad \xi = \tan^{-1} \frac{2cz}{r^2 + z^2 - c^2}, \quad (3)$$

taking the origin at the middle point of the line between the limiting or inverse points. The position of these points is determined in what follows.

The transformed equation is

$$\frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \xi^2} - \frac{\sinh \eta}{\cosh \eta - \cos \xi} \frac{\partial V}{\partial \eta} + \frac{\cosh \eta \cos \xi - 1}{\sin \xi (\cosh \eta - \cos \xi)} \frac{\partial V}{\partial \xi} = 0. \quad (4)$$

Equation (4) is transformed by the substitution $V = U\sqrt{2(\cosh \eta - \cos \xi)}$ into

$$\frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \xi^2} + \cos \xi \frac{\partial U}{\partial \xi} - \frac{U}{4} = 0. \quad (5)$$

Substituting $U = PQ$, where P is a function of η alone and Q a function of ξ alone, we get the ordinary equations

$$\frac{d^2 P}{d\eta^2} - (m + \frac{1}{2})^2 P = 0, \quad \frac{d^2 Q}{d\xi^2} + \cot \xi \frac{dQ}{d\xi} + m(m+1)Q = 0. \quad (6)$$

The particular solutions of (6) are used in the manner described below to build up a solution to fit the given conditions, which are that $V = V_1$ when $\eta = \alpha$, and $V = V_2$ when $\eta = -\beta$.

The Inverse Points of Two Circles.—By a well-known theorem in geometry, the locus of a point which moves so that the ratio of its distance from two fixed points is constant, is a circle, and if C be the center of the circle and A and B the two fixed points, A , B and C are collinear and

$$CA \times CB = a^2.$$

The points A and B are called inverse or limiting points with respect to the circle. Obviously any circle has an infinite number of pairs of inverse points in a given straight line through the center. Two given circles will have one pair of inverse points in common.

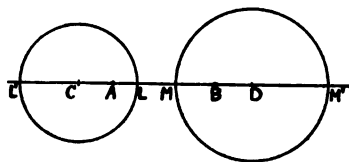


Fig. 1.

Let A and B be the inverse points, C and D the centers of the circles, L , L' ,

M , M' the points where the line of centers cuts the circles (Fig. 1).

Let $CL = a$, $MD = b$.

Let $AB = 2c$, then $CB \cdot CA = a^2$, or $CB(CB - 2c) = a^2$.

Solving for CB ,

$$CB = c + \sqrt{a^2 + c^2}. \quad (1)$$

Also,

$$DB(DB + 2c) = b^2.$$

Solving for DB ,

$$DB = -c + \sqrt{b^2 + c^2}. \quad (2)$$

If $d = CB$, we get by adding (1) and (2)

$$d = \sqrt{a^2 + c^2} + \sqrt{b^2 + c^2}. \quad (3)$$

Solving (3) for c ,

$$c^2 = \frac{4}{a^2} (d + a + b)(d + a - b)(d + b - a)(d - a - b). \quad (4)$$

Equations (1), (2) and (4) locate the point B .

To find position of A we have

$$CA = CB - 2c = \sqrt{a^2 + c^2} - c, \quad (5)$$

$$DA = d - AC = \sqrt{b^2 + c^2} + c. \quad (6)$$

When $a = b$, $d = 2\sqrt{a^2 + c^2}$, and

$$c = \sqrt{\frac{d^2}{4} - a^2}. \quad (7)$$

The Equations of the First Transformation of Laplace's Equation.—The potential at any point satisfies Laplace's equation which in cylindrical coördinates is

$$\frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{z} \frac{\partial V}{\partial z} = 0, \quad (8)$$

where z is the distance of the point from the axis or line of centers, and r is the distance of the point from any fixed plane normal to the axis.

Let r_1, r_2 denote the distances of the point from the inverse points of the sheres, e^η the ratio of these distances, ξ the angle between them. Then (Fig. 2)

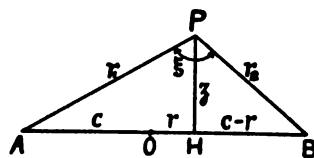


Fig. 2.

$$e^\eta = \frac{r_2}{r_1} \quad \text{or} \quad \eta = \log_e \left(\frac{r_2}{r_1} \right) \quad (9)$$

and

$$4c^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos \xi$$

or

$$\cos \xi = \frac{r_1^2 + r_2^2 - 4c^2}{2r_1r_2}. \quad (10)$$

Taking origin at middle point of AB ,

$$r_1^2 = z^2 + (c + r)^2, \quad r_2^2 = z^2 + (c - r)^2. \quad (11)$$

From (11)

$$r_1^2 + r_2^2 - 4c^2 = 2(z^2 + r^2 - c^2), \quad (12)$$

$$\begin{aligned} r_1^2 r_2^2 &= (z^2 + r^2 + c^2)^2 - 4r^2 c^2 \\ &= (z^2 + r^2 - c^2)^2 + 4c^2 z^2. \end{aligned} \quad (13)$$

Then by (10),

$$\cos \xi = \frac{z^2 + r^2 - c^2}{\sqrt{(z^2 + r^2 - c^2)^2 + 4c^2 z^2}}$$

and

$$\tan \xi = \frac{2cz}{z^2 + r^2 - c^2}. \quad (14)$$

From (9)

$$\eta = \frac{1}{2} \log \left[\frac{z^2 + (c+r)^2}{z^2 + (c-r)^2} \right]. \quad (15)$$

From (14) and (15)

$$\frac{r^2 + z^2 - c^2}{2cz} = \cot \xi, \quad \text{and} \quad \frac{r^2 + z^2 + c^2}{2cr} = \coth \eta. \quad (16)$$

From (16)

$$2cz \cot \xi = r^2 + z^2 - c^2, \quad \text{and} \quad 2cr \coth \eta = r^2 + c^2 + z^2. \quad (17)$$

Subtracting and removing factor $2c$,

$$r(\coth \eta - \cot \xi) = c. \quad (18)$$

From (17)

$$r^2 \operatorname{cosech}^2 \eta = \frac{(r^2 + c^2 + z^2)^2 - 4c^2 r^2}{4c^2}, \quad z^2 \operatorname{cosec}^2 \xi = \frac{(r^2 + z^2 + c^2) - 4c^2 r^2}{4c^2}.$$

Therefore

$$r \operatorname{cosech} \eta = z \operatorname{cosec} \xi. \quad (19)$$

Solving (18) and (19),

$$r = \frac{c \sinh \eta}{\cosh \eta - \cos \xi}, \quad \text{and} \quad z = \frac{c \sin \xi}{\cosh \eta - \cos \xi}. \quad (20)$$

Partial Derivatives of η and ξ with Respect to r and z .—The functions η and ξ are conjugate functions as will be seen when the derivatives are written.

$$\frac{\partial \eta}{\partial r} = \frac{2c(z^2 - r^2 + c^2)}{(z^2 + r^2 + c^2)^2 - 4r^2 c^2}, \quad \frac{\partial \xi}{\partial z} = \frac{-2c(z^2 - r^2 + c^2)}{(z^2 + r^2 + c^2)^2 - 4r^2 c^2}, \quad (21)$$

$$\frac{\partial \eta}{\partial z} = \frac{-4c z r}{(r^2 + z^2 + c^2)^2 - 4r^2 c^2}, \quad \frac{\partial \xi}{\partial r} = \frac{-4c z r}{(r^2 + z^2 + c^2)^2 - 4r^2 c^2}. \quad (22)$$

Equations (21) and (22) show that

$$\frac{\partial \eta}{\partial r} = -\frac{\partial \xi}{\partial z} \quad \text{and} \quad \frac{\partial \eta}{\partial z} = \frac{\partial \xi}{\partial r}. \quad (23)$$

From (23) we easily obtain

$$\frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial z^2} = 0, \quad \text{and} \quad \frac{\partial^2 \xi}{\partial r^2} + \frac{\partial^2 \xi}{\partial z^2} = 0. \quad (24)$$

Also

$$\left(\frac{\partial \eta}{\partial r}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2 = \left(\frac{\partial \xi}{\partial r}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2 = \frac{4c^2}{(r^2 + z^2 + c^2)^2 - 4r^2c^2}. \quad (25)$$

First Transformation of Laplace's Equation.—The relations (24) and (25) make the transformation quite easy.

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial \eta} \cdot \frac{\partial \eta}{\partial z} + \frac{\partial V}{\partial \xi} \cdot \frac{\partial \xi}{\partial z}, \quad (26)$$

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial \eta} \cdot \frac{\partial \eta}{\partial r} + \frac{\partial V}{\partial \xi} \cdot \frac{\partial \xi}{\partial r}, \quad (27)$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial z}\right)^2 + \frac{\partial V}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial z^2} + \frac{\partial^2 V}{\partial \xi^2} \cdot \left(\frac{\partial \xi}{\partial z}\right)^2 + \frac{\partial V}{\partial \xi} \cdot \frac{\partial^2 \xi}{\partial z^2}, \quad (28)$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{\partial^2 V}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial r}\right)^2 + \frac{\partial V}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 V}{\partial \xi^2} \cdot \left(\frac{\partial \xi}{\partial r}\right)^2 + \frac{\partial V}{\partial \xi} \cdot \frac{\partial^2 \xi}{\partial r^2}. \quad (29)$$

Adding (28) and (29) and using (24) and (25)

$$\frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial z^2} = \frac{4c^2}{(r^2 + z^2 + c^2)^2 - 4c^2r^2} \cdot \left\{ \frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \xi^2} \right\}. \quad (30)$$

From (26) and (21) and (22)

$$\frac{1}{z} \frac{\partial V}{\partial z} = \frac{\partial V}{\partial \eta} \left(\frac{-4cr}{(r^2 + z^2 + c^2) - 4c^2r^2} \right) + \frac{\partial V}{\partial \xi} \left(\frac{-2c(r^2 - z^2 - c^2)}{z[(r^2 + z^2 + c^2)^2 - 4c^2r^2]} \right). \quad (31)$$

Adding (30) and (31) and removing factor $4c^2/(r^2 + z^2 + c^2)^2 - 4c^2r^2$,

$$\frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \xi^2} - \frac{r}{c} \cdot \frac{\partial V}{\partial \eta} + \frac{r^2 - z^2 - c^2}{2cz} \cdot \frac{\partial V}{\partial \xi} = 0. \quad (32)$$

It remains to substitute the value of r/c and of $r^2 - z^2 - c^2/2cz$ in terms of η and ξ . The value of r and of z are given by (20).

$$\frac{r}{c} = \frac{\sinh \eta}{\cosh \eta - \cos \xi}, \quad \frac{r^2 - z^2 - c^2}{2cz} = \frac{\cosh \eta \cos \xi - 1}{\sin \xi (\cosh \eta - \cos \xi)}.$$

Then (32) becomes

$$\begin{aligned} \frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \xi^2} - \frac{\sinh \eta \sin \xi}{\sin \xi (\cosh \eta - \cos \xi)} \frac{\partial V}{\partial \eta} \\ + \frac{\cosh \eta \cos \xi - 1}{\sin \xi (\cosh \eta - \cos \xi)} \frac{\partial V}{\partial \xi} = 0. \end{aligned} \quad (33)$$

Second Transformation.—Let $V = U\sqrt{2(\cosh \eta - \cos \xi)}$. For convenience of manipulation put $\sqrt{2(\cosh \eta - \cos \xi)} = F$.

$$\frac{\partial F}{\partial \eta} = \frac{\sinh \eta}{F}, \quad \frac{\partial^2 F}{\partial \eta^2} = \frac{F^2 \cosh \eta - \sinh^2 \eta}{F^3}, \quad (34)$$

$$\frac{\partial F}{\partial \xi} = \frac{\sin \xi}{F}, \quad \frac{\partial^2 F}{\partial \xi^2} = \frac{F^2 \cos \xi - \sin^2 \xi}{F^3}, \quad (35)$$

$$\frac{\partial V}{\partial \eta} = F \frac{\partial U}{\partial \eta} + U \frac{\partial F}{\partial \eta} = F \frac{\partial U}{\partial \eta} + \frac{U \sinh \eta}{F}, \quad (36)$$

$$\frac{\partial V}{\partial \xi} = F \frac{\partial U}{\partial \xi} + U \frac{\partial F}{\partial \xi} = F \frac{\partial U}{\partial \xi} + \frac{U \sin \xi}{F}, \quad (37)$$

$$\frac{\partial^2 V}{\partial \eta^2} = F \frac{\partial^2 U}{\partial \eta^2} + \frac{2 \sinh \eta}{F} \frac{\partial U}{\partial \eta} + U \left\{ \frac{F^2 \cosh \eta - \sinh^2 \eta}{F^3} \right\}, \quad (38)$$

$$\frac{\partial^2 V}{\partial \xi^2} = F \frac{\partial^2 U}{\partial \xi^2} + \frac{2 \sin \xi}{F} \frac{\partial U}{\partial \xi} + U \left\{ \frac{F^2 \cos \xi - \sin^2 \xi}{F^3} \right\}. \quad (39)$$

Substituting (36), (37), (38), (39) in (33) and collecting coefficients

$$F \left(\frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \xi^2} \right) + \frac{2 \sinh \eta - 2 \sin \xi}{F} \frac{\partial U}{\partial \eta} + \frac{2(\sin^2 \xi + \cosh \eta \cos \xi - 1)}{F \sin \xi} \frac{\partial U}{\partial \xi} \\ + U \left\{ \frac{F^2 (\cosh \eta + \cos \xi) - 2 \sinh^2 \eta - \sin^2 \xi + 2 (\cosh \eta - \cos \xi) - 2}{F^3} \right\} = 0,$$

which reduces to

$$\frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \xi^2} + \cot \xi \frac{\partial U}{\partial \xi} - \frac{U}{4} = 0. \quad (40)$$

Third Transformation.—If we put $U = RQ$, where R is a function of η alone and Q a function of ξ alone, we get

$$Q \frac{d^2 R}{d\eta^2} + R \frac{d^2 Q}{d\xi^2} + \cot \xi R \frac{dQ}{d\xi} - \frac{RQ}{4} = 0.$$

Dividing by RQ and separating functions of η from those of ξ .

$$\left(\frac{1}{R} \frac{d^2 R}{d\eta^2} - \frac{1}{4} \right) = - \frac{1}{Q} \left(\frac{d^2 Q}{d\xi^2} + \cot \xi \frac{dQ}{d\xi} \right). \quad (41)$$

The left member of (41) does not contain ξ , and the right member does not contain η , hence they must each be equal to a constant which it is convenient to denote by $m(m+1)$, m being any positive integer.

Then

$$\frac{d^2 R}{d\eta^2} - (m^2 + m + \frac{1}{2})R = 0, \text{ or } \frac{d^2 R}{d\eta^2} - (m + \frac{1}{2})^2 R = 0. \quad (42)$$

Particular solutions of this equation are $e^{(m+1/2)\eta}$ and $e^{-(m+1/2)\eta}$. Also

$$\frac{d^2 Q}{d\xi^2} + \cot \xi \frac{dQ}{d\xi} + m(m+1)Q = 0. \quad (43)$$

Equation (43) is satisfied by $P_m(\cos \xi)$ when m is a positive integer, $P_m(\cos \xi)$ denoting a zonal harmonic or Legendre function. (See Byerly's Fourier Series and Spherical Harmonics, Chapter I.) Then

$$U = \sum_0^\infty A_m R_{1m} P_m(\cos \xi) + \sum_0^\infty B_m R_{2m} P_m(\cos \xi) \quad (44)$$

and

$$V = U\sqrt{2(\cosh \eta - \cos \xi)}, \quad (45)$$

where R_{1m} and R_{2m} are particular solutions of (43) which fulfill certain requirements, and A_m and B_m are constants in certain special expansions. The requirements are the following:

U shall reduce to $\frac{V_1}{\sqrt{2(\cosh \alpha - \cos \xi)}}$ when $\eta = \alpha$ for all values of ξ ;

U shall reduce to $\frac{V_2}{\sqrt{2(\cosh \beta - \cos \beta)}}$ when $\eta = -\beta$ for all values of ξ ;

R_{1m} is to be determined so that its value is zero when $\eta = -\beta$ and unity when $\eta = \alpha$;

R_{2m} is to be determined so that its value is zero when $\eta = \alpha$ and unity when $\eta = -\beta$.

The particular solutions $e^{(m+1/2)\eta}$ and $e^{-(m+1/2)\eta}$ of the equation

$$\frac{d^2 P}{d\eta^2} - (m + \frac{1}{2})^2 P = 0$$

may be multiplied by any constants, and added or subtracted and the results will still be solutions. Then $e^{-(m+1/2)\alpha} \cdot e^{+(m+1/2)\eta}$ and $e^{(m+1/2)\alpha} \times e^{-(m+1/2)\eta}$ are solutions. These are the same as $e^{-(m+1/2)(\alpha-\eta)}$ and $e^{(m+1/2)(\alpha-\eta)}$. Then $\frac{1}{2}\{e^{+(m+1/2)(\alpha-\eta)} - e^{-(m+1/2)(\alpha-\eta)}\}$ is a solution which vanishes for $\eta = \alpha$. It is identical with $\sinh(m + \frac{1}{2})(\alpha - \eta)$. The function

$$\frac{\sinh(m + \frac{1}{2})(\alpha - \eta)}{\sinh(m + \frac{1}{2})(\alpha + \beta)}$$

is equal to unity when $\eta = -\beta$. In like fashion

$$\frac{\sinh(m + \frac{1}{2})(\beta + \eta)}{\sinh(m + \frac{1}{2})(\alpha + \beta)}$$

vanishes when $\eta = -\beta$ and is equal to unity when $\eta = \alpha$. We have then

$$R_{1m} = \frac{\sinh(m + \frac{1}{2})(\beta + \eta)}{\sinh(m + \frac{1}{2})(\alpha + \beta)} \quad \text{and} \quad R_{2m} = \frac{\sinh(m + \frac{1}{2})(\alpha - \eta)}{\sinh(m + \frac{1}{2})(\alpha + \beta)}.$$

When $\eta = \alpha$, we have

$$\frac{V_1}{\sqrt{2}(\cosh \alpha - \cos \xi)} = \sum_0^{\infty} A_m P_m(\cos \xi),$$

because R_{1m} is unity.

To find the values of A_m , the function $\{2(\cosh \alpha - \cos \xi)\}^{-1/2}$ must be expanded in a series of zonal harmonics.

$P_m(x)$ is defined as the coefficient of x^m in the expansion of $(1 - 2xz + z^2)^{-1/2}$. The function $\{2(\cosh \alpha - \cos \xi)\}^{-1/2}$ is easily thrown into this form, for we have $2 \cosh \alpha = e^\alpha + e^{-\alpha}$, and

$$\{e^\alpha + e^{-\alpha} - 2 \cos \xi\}^{-1/2} = e^{-\alpha/2} \{1 - 2e^{-\alpha} \cos \xi + e^{-2\alpha}\}^{-1/2}.$$

Then $P_m(\cos \xi)$ is the coefficient of $e^{-m\alpha}$ in the expansion of

$$\{1 - 2e^{-\alpha} \cos \xi + e^{-2\alpha}\}^{-1/2}.$$

Consequently

$$\begin{aligned} \{2(\cosh \alpha - \cos \xi)\}^{-1/2} &= e^{-\alpha/2} \sum_0^{\infty} e^{-m\alpha} P_m(\cos \xi), \\ &= \sum_0^{\infty} e^{-(m+1/2)\alpha} P_m(\cos \xi). \end{aligned}$$

Obviously,

$$A_m = V_1 e^{-(m+1/2)\alpha}$$

and

$$B_m = V_2 e^{-(m+1/2)\beta}.$$

Substituting the values of A_m , B_m , R_{1m} and R_{2m} in the formula

$$V = \sum_0^{\infty} A_m R_{1m} P_m(\cos \xi) + \sum_0^{\infty} B_m R_{2m} P_m(\cos \xi)$$

we get the formula at the beginning of the paper.

FORMULÆ FOR SPECIAL VALUES OF V_1 AND V_2 .

Equal Spheres, Potentials $V_1/2$ and $-V_1/2$.—Here $\alpha = \beta$, then

$$V = \frac{V_1}{2} \sqrt{2 (\cosh \eta - \cos \xi)} \sum_0^{\infty} \frac{\sinh (m + \frac{1}{2})(\alpha + \eta) - \sinh (m + \frac{1}{2})(\alpha - \eta)}{\sinh (m + \frac{1}{2})(\alpha + \alpha)} \cdot P_m (\cos \xi) e^{-(m+1/2)\alpha}.$$

Since $\sinh A - \sinh B = 2 \sinh \frac{1}{2}(A - B) \cosh \frac{1}{2}(A + B)$, the numerator becomes $2 \sinh \left(\frac{2m + 1}{2} \right) \eta \cosh (m + \frac{1}{2})\alpha$, and then

$$V = \frac{V_1}{2} \sqrt{2 (\cosh \eta - \cos \xi)} \sum_0^{\infty} \frac{\sinh (m + \frac{1}{2})\eta}{\sinh (m + \frac{1}{2})\alpha} \cdot P_m (\cos \xi) e^{-(m+1/2)\alpha}. \quad (46)$$

It is easily seen that this reduces to $V_1/2$, when $\eta = \alpha$.

Equal Spheres, Potentials V_1 and 0.—Again $\alpha = \beta$, and

$$V = V_1 \{2 (\cosh \eta - \cos \xi)\}^{1/2} \sum_0^{\infty} \frac{\sinh (m + \frac{1}{2})(\alpha + \eta)}{\sinh (m + \frac{1}{2})(2\alpha)} P_m (\cos \xi) e^{-(m+1/2)\alpha}. \quad (47)$$

These appear to be the only cases of interest to physicists.

II. THE ELECTROSTATIC FORCE OR VOLTAGE GRADIENT.

At any point in the line of centers, between the spheres, $\cos \xi = -1$. Then, for the voltage gradient G , we have

$$G = \frac{\partial V}{\partial \eta} \cdot \frac{\partial \eta}{\partial r}. \quad (48)$$

In this special case $z = 0$, and

$$\eta = \log \frac{c + r}{c - r},$$

by (3). Then

$$\frac{\partial \eta}{\partial r} = \frac{2c}{c^2 - r^2}. \quad (49)$$

Let x = distance between surfaces, then $x + 2a = \sqrt{a^2 + c^2}$, by (7). Then

$$2c = \sqrt{x^2 + 4ax},$$

and

$$\frac{\partial \eta}{\partial r} = \frac{\sqrt{x^2 + 4ax}}{ax},$$

since $r = x/2$ at surface. Also

$$e^{\alpha} = \frac{\sqrt{x^2 + 4ax} + x}{\sqrt{x^2 + 4ax} - x},$$

whence

$$\frac{\sqrt{x^2 + 4ax}}{ax} = \frac{1}{a} \coth \frac{\alpha}{2}.$$

In the case of equal spheres, potentials $+V_1/2$ and $-V_1/2$, we have

$$V = \frac{V_1}{2} \sqrt{2 (\cosh \eta - \cos \xi)} \sum_0^{\infty} \frac{\sinh (m + \frac{1}{2}) \eta}{\sinh (m + \frac{1}{2}) \alpha} P_m (\cos \xi) e^{-(m+1/2)\alpha},$$

$$\frac{\partial V}{\partial \eta} = \frac{V_1}{2} \left\{ \sqrt{2 (\cosh \eta - \cos \xi)} \sum_0^{\infty} \frac{\cosh (m + \frac{1}{2}) \eta}{\sinh (m + \frac{1}{2}) \alpha} \cdot (m + \frac{1}{2}) P_m (\cos \xi) e^{-(m+1/2)\alpha} \right.$$

$$\left. + \frac{\sinh \eta}{\sqrt{2 (\cosh \eta - \cos \xi)}} \sum_0^{\infty} \frac{\sinh (m + \frac{1}{2}) \eta}{\sinh (m + \frac{1}{2}) \alpha} P_m (\cos \xi) e^{-(m+1/2)\alpha} \right\}.$$

When $\eta = \alpha$, and $\cos \xi = -1$,

$$\left(\frac{\partial V}{\partial \eta} \right)_{\eta=\alpha} = \frac{V_1}{2} \left\{ \sinh \frac{\alpha}{2} \sum_0^{\infty} P_m (-1) e^{-(m+1/2)\alpha} \right.$$

$$\left. + 2 \cosh \frac{\alpha}{2} \sum_0^{\infty} (m + \frac{1}{2}) \coth (m + \frac{1}{2}) \alpha P_m (-1) e^{-(m+1/2)\alpha} \right\}.$$

$$\sum_0^{\infty} P_m (-1) e^{-(m+1/2)\alpha} = e^{-\alpha/2} (1 - e^{-\alpha} + e^{-2\alpha} - e^{-3\alpha} + \dots)$$

$$= e^{-\alpha/2} (1 + e^{-\alpha})^{-1} = \frac{1}{2 \cosh \frac{\alpha}{2}}.$$

Then

$$\left(\frac{\partial V}{\partial \eta} \right)_{\eta=\alpha} = V \left\{ \frac{1}{4} \tanh \frac{\alpha}{2} + \cosh \frac{\alpha}{2} \left[\frac{1}{2} e^{-\alpha/2} \coth \frac{\alpha}{2} - \frac{3}{2} e^{-3\alpha/2} \coth \frac{3\alpha}{2} \right. \right.$$

$$\left. \left. + \frac{5}{2} e^{-5\alpha/2} \coth \frac{5\alpha}{2} - \frac{7}{2} e^{-7\alpha/2} \coth \frac{7\alpha}{2} + \dots \right] \right\}$$

and

$$G = \frac{V_1}{a} \left\{ \frac{1}{4} + \coth \frac{\alpha}{2} \cosh \frac{\alpha}{2} \left[\frac{1}{2} e^{-\alpha/2} \coth \frac{\alpha}{2} - \frac{3}{2} e^{-3\alpha/2} \coth \frac{3\alpha}{2} + \dots \right] \right\}. \quad (50)$$

The series $\sum_0^{\infty} (m + \frac{1}{2}) e^{-(m+1/2)\alpha} \coth (m + \frac{1}{2}) \alpha$ is highly convergent

for large values of α , but very slowly convergent for small values of α .

In practice the values of x/a are given, the values of α calculated, and the values of the functions taken out of tables of hyperbolic and exponential functions.

We have

$$\alpha = \log_e \frac{\sqrt{x^2 + 4ax} + x}{\sqrt{x^2 + 4ax} - x}.$$

For example if the value of x/a is $\frac{1}{2}$, we get $\alpha = \log_e 2 = 2.3025 \log_{10} 2 = 0.69314$. When x/a is very large, α is very large, and then we have, very nearly, $\coth \alpha/2 = 1$, $\coth 3\alpha/2 = 1$, etc., and $\cosh \alpha/2 = \frac{1}{2}e^{\alpha/2}$, and G reduces to

$$\begin{aligned} G &= \frac{V_1}{a} \left\{ \frac{1}{2} + \frac{1}{4} [1 - 3e^{-\alpha} + 5e^{-2\alpha} - 7e^{-3\alpha} + \dots] \right\} \\ &= \frac{V_1}{a} \left\{ \frac{1}{4} + \frac{1}{4} \left[\frac{1 - e^{-\alpha}}{(1 + e^{-\alpha})^2} \right] \right\} = \frac{V}{x} \left\{ \frac{1}{2} \left(\frac{x}{a} + 1 \right) \right\}. \end{aligned} \quad (51)$$

When the equal spheres have potentials V_1 and 0, the potential is

$$V = V_1 \sqrt{2 (\cosh \eta - \cos \xi)} \sum_0^{\infty} \frac{\sinh (m + \frac{1}{2})(\alpha + \eta)}{\sinh (m + \frac{1}{2})(\alpha + \beta)} P_m (\cos \xi) e^{-(m+1/2)\alpha}.$$

Then

$$\begin{aligned} \frac{\partial V}{\partial \eta} &= V_1 \left\{ \frac{\sinh \eta}{\sqrt{2 (\cosh \eta - \cos \xi)}} \sum_0^{\infty} \frac{\sinh (m + \frac{1}{2})(\alpha + \eta)}{\sinh (m + \frac{1}{2})(2\alpha)} P_m' (\cos \xi) e^{-(m+1/2)\alpha} \right. \\ &\quad \left. + \sqrt{2 (\cosh \eta - \cos \xi)} \sum_0^{\infty} \frac{(m + \frac{1}{2}) \cosh (m + \frac{1}{2})(\alpha + \eta)}{\sinh (m + \frac{1}{2})2\alpha} P_m (\cos \xi) e^{-(m+1/2)\alpha} \right\}. \end{aligned}$$

When $\eta = \alpha$, and $\cos \xi = -1$,

$$\begin{aligned} \left(\frac{\partial V}{\partial \eta} \right)_{\eta=\alpha} &= V_1 \left\{ \sinh \frac{\alpha}{2} \sum_0^{\infty} P_m (-1) e^{-(m+1/2)\alpha} \right. \\ &\quad \left. + 2 \cosh \frac{\alpha}{2} \sum_0^{\infty} (m + \frac{1}{2}) P_m (-1) e^{-(m+1/2)\alpha} \coth (2m + 1)\alpha \right\} \\ &= \frac{V}{2} \left\{ \tanh \frac{\alpha}{2} + 4 \cosh \frac{\alpha}{2} \left[\frac{1}{2} e^{-\alpha/2} \coth \alpha - \frac{3}{2} e^{-3\alpha/2} \coth 3\alpha + \dots \right] \right\}, \end{aligned}$$

and

$$G = \frac{V}{a} \left\{ \frac{1}{2} + 2 \coth \frac{\alpha}{2} \cosh \frac{\alpha}{2} \left[\frac{1}{2} e^{-\alpha/2} \coth \alpha - \frac{3}{2} e^{-3\alpha/2} \coth 3\alpha + \dots \right] \right\}.$$

This series like the other is rapidly convergent for large values of α , but very slowly for small values. The limiting value of G for large values is $\frac{V}{x} \left\{ \frac{x}{a} + \frac{a}{x} \right\}$.

AN ATTEMPT TO DETECT POSSIBLE CHANGES IN WEIGHT OR MOMENTUM EFFECTS ON CHARGING A CONDENSER.

BY P. G. AGNEW AND W. C. BISHOP.

THERE is no reason to suppose that the weight of a body is affected by its state of electrification in great enough a degree to be detected by the methods of direct weighing at present available. Yet however improbable such an effect may seem, the question cannot be settled without an appeal to experiment. Similar investigations have been made to detect possible changes in weight due to temperature¹ and to magnetization,² which have, as was anticipated, given negative results.

Southerns has attacked the problem by inducing opposite charges at the ends of the beam of a balance (which was earthed) by means of an external electrostatic field and noting the difference in readings on reversing the external field.³

A small but definite residual appeared in his results which he was unable to account for by experimental errors. He makes it clear, however, that he does not regard it as an established experimental fact that the weight of a body depends upon its electrification, as the sensitivity had been pushed to an extreme, and consequently the small residual may have been due to a systematic error. In fact, unless the result is confirmed by further work this must remain the most probable explanation. He says, "Whether the effect be due to a modification of gravity or not, the experiments appear to show that a body, positively charged with 20 electrostatic units of electricity, behaves as though it were heavier than the same body negatively charged, by an amount of the order of .0007 mgm."

CHANGE OF WEIGHT.

It seemed worth while to make some weighings on a condenser with a good balance to see whether any difference in weight could be detected between the charged and the discharged conditions. The experiment is analogous but not equivalent to Southerns's since both plates of the condenser are in this case necessarily on the same pan of the balance and hence if the positive plate loses in weight the same amount that the

¹ Southerns, *Proc. Roy. Soc.*, A 78, p. 392, 1906.

² Lloyd, *Terrestrial Magnetism*, 14, p. 67, 1909.

³ Southerns, *Proc. Camb. Phil. Soc.*, 15, p. 352, 1909.

negative plate gains, or vice versa, it could not be detected. There is the advantage, however, that very much greater charges may be used in the light paper condensers now available. And further, the external electrostatic field may be made extremely small, thus minimizing the chance of extraneous effects entering.

Since all the results are negative, the work will be described but briefly.

After preliminary work with other types, the final weighings were made with five 2 mf. condensers of the Mansbridge type. They were joined together and shielded as a single condenser, a hole just sufficient for the leading in wire being left in the tin foil shield to which the one plate was connected. In the final weighings, this plate and the metallic parts of the balance, including the levelling screws, were earthed irrespective of the condition of charge of the condenser. In order to be able to charge and discharge the condenser without arresting the balance, a flexible connection was made by two 0.02 mm. silver wires about 6 cm. long. These gave no trouble whatever in decreasing the sensitivity of the balance, which was a Becker type designed for a load of a kilogram. To avoid possible heating of the leading-in wires the condenser was always charged and discharged through a resistance of 500 ohms. Electromagnetic control of the amplitude of the swing was found very convenient. For this purpose iron was used as a part of the counterweight and a small electromagnet placed under the balance case. The amplitude could easily be controlled to a few tenths of a mm.

In taking the observations 5 turning points were read for each rest point. The readings were taken as follows. With the positive pole of the battery connected to earth 5 rest points were taken, the condenser being charged and discharged alternately between rest points, and the whole then repeated with the negative pole of the battery to earth. Such a set of observations will be referred to as a group.

After a considerable number of preliminary difficulties were overcome no group of readings was made under satisfactory conditions which showed a greater difference than 0.1 mm. on the scale between the average of the 6 rest points taken with the condenser charged and the 4 rest points taken with the condenser uncharged. This corresponds to 0.015 mg.

The average with the + pole of the battery to earth was never quite the same as that with the - pole to earth. This difference, which was nearly always less than 0.1 mm. on the scale, differed in direction from evening to evening, but was usually in the same direction throughout a given evening. It was probably due to small charges within the balance case acting on the leading-in wires. It was decreased by placing radio-

active matter in the case, and by leaving the balance undisturbed for some time before taking readings, but it never entirely disappeared.

There seemed to be no particular advantage in taking any very large number of observations, so on an evening when conditions were the very best four groups of readings were taken in a continuous series during which time neither observer so much as moved from his position in order that no disturbing factor might enter. No reading was lost and none have been omitted in calculating the results given below. The final average depends upon 4 groups during which 40 rest points or 200 turning points were taken.

Capacity of condenser.....	10 mf.
Voltage.....	240
Quantity (coulombs).....	$2.4 \cdot 10^{-3}$
Quantity (electrostatic units).....	$7.2 \cdot 10^8$
Weight of condensers.....	930 g.
Scale distance.....	3.1 meters
Period of balance.....	45 sec.
Sensitivity, 1 cm. of scale = ...	1.5 mg.
Av. rest point, condensers charged.....	84.253
Av. rest point, condensers discharged.....	84.255
Difference = ...	0.002 cm.
Apparent increase in weight = ...	0.003 mg.

It is seen that the actual average difference in weight found is one part in 300,000,000, but it is probably better to estimate the accuracy as somewhat lower, say in round numbers, 0.01 mg. or one part in 100,000,000

As already pointed out our negative result does not necessarily conflict with the apparent small positive effect found by Southern, since his experiment attempted to distinguish between the weight of a body when positively and when negatively charged. If the positive plate increases and the negative plate decreases by an exactly equal amount Southern's experiment would give a positive result while a condenser would not change its weight upon charge or discharge. But if the plates change in weight by amounts which are not equal and opposite, the weight of the whole condenser must, of course, be changed. If we assume the small difference found by Southern to be a true effect and to be proportional to the charge, the difference in weight per electrostatic unit would be $3.5 \cdot 10^{-8}$ g., or 100 g. per coulomb. According to the present experiment the change per electrostatic unit is less than $1.4 \cdot 10^{-12}$ g. or less than 4 mg. per coulomb.

The electromagnetic mass is of course a quantity of an entirely different order of magnitude than that which may be reached by the modern precision balance; w/c^2 in the present case is but the billionth part of the actual residual found (0.003 mg.).

MOMENTUM.

Inasmuch as the same arrangement of apparatus sufficed for the purpose, an attempt was made to detect possible momentum effects upon charge and discharge of the condenser. A condenser standing upright on the scale pan and charged from above might receive a kick on charge and an opposite kick on discharge and, however improbable, it was decided also to look for the possibility of a kick which would be in the same direction on both charge and discharge.

The method used was to time the charge and discharge so that any momentum effect would tend to act cumulatively in changing the amplitude of swing. In order to look for the first kind of an effect, the balance was given a small amplitude of swing and the condenser charged during 10 swings to the right, and discharged during the 10 swings to the left, and then the process reversed for 10 complete swings. It is evident that if there is such an effect as looked for the first half of the process ought to tend to either increase or to decrease the amplitude, while the opposite effect should take place during the second half.

A group of actual readings will make the matter clear and at the same time show how the effect of damping can be eliminated.

March 15, 1912—to Earth.

Turning Point.

Double Amplitude.

8.91	.80	} Charged on increasing and discharged on decreasing swings.
8.11	.79	
8.90	.78	
8.12	.77	
8.89	.76	
8.13	.75	
8.88	.70	
8.18	.69	
8.87	.67	
8.20	.68	
8.88		
8.22	.66	} Charged on decreasing and discharged on increasing swings.
8.88	.65	
8.23	.64	
8.87	.63	
8.24	.59	
8.83	.59	
8.24	.57	
8.81	.56	
8.25	.55	
8.80	.52	
8.28		

It was found that for the small amplitudes the damping was small enough so that a sensibly straight line was obtained by plotting amplitude against time. Hence if there is no true momentum effect we should get a constant by adding the first 10 amplitudes to the second 10 amplitudes taken in reversed order, that is, adding .80 to .52, .79 to .55 and so on,

but if there is a momentum effect then these numbers should either constantly increase or constantly decrease. Such an addition gives

1.32
1.34
1.34
1.34
1.35
1.34
1.33
1.33
1.32
1.34

It is seen that these numbers are constant within the errors of observation. In a careful examination of all the runs taken under favorable conditions (after various disturbing factors had been removed), no tendency either to increase or to decrease the amplitude could be detected. A conservative estimate of the amount of change in amplitude which could have been readily and unmistakably detected is 0.04 cm. for the group of 10 swings or 0.004 cm. per swing.

In order to interpret this change in amplitude in C.G.S. units of momentum, we must consider the constants of the balance when swinging through a small angle. It may easily be shown that a given change in amplitude means a given change in momentum, no matter what the amplitude is, the maximum angular momentum being proportional to the square of the amplitude, just as the maximum kinetic energy is proportional to the square of the amplitude.

Let A = amplitude of swing (radians).

A_0 = amplitude of swing (cm.).

f = force in dynes necessary to give 1 cm. deflection when applied to a pan.

I = equivalent moment of inertia of moving parts.

K = restoring torque.

l = length of balance arm.

M = momentum.

s = scale distance.

T = complete period of balance.

U = angular momentum at middle of swing.

θ = angle of displacement.

Then

$$T = 2\pi \sqrt{\frac{I}{K}}, \quad (1)$$

$$\theta = A \cos \frac{2\pi}{T} t$$

and

$$U = I\dot{\theta}_{\max} = \frac{2\pi IA}{T}. \quad (2)$$

Eliminating I from (1) and (2)

$$U = \frac{KT}{2\pi} A,$$

which shows that the maximum angular momentum is proportional to the amplitude. Now if a small momentum ΔM is added to one pan the angular momentum is changed by

$$\begin{aligned} \Delta U &= l\Delta M, \\ \therefore \Delta M &= \frac{KT}{2\pi l} \Delta A. \end{aligned} \quad (3)$$

Since deflections are read by a mirror 1 cm. is equivalent to $1/2s$ radians. Hence the torque per radian is

$$K = 2sfl.$$

We also have

$$\begin{aligned} A &= \frac{A_0}{2s}, \\ \therefore \Delta M &= \frac{fT}{2\pi} \Delta A_0. \end{aligned}$$

In the present case $f = 1.5$, $T = 45$, $\Delta A_0 = 0.001$ (since a difference of 0.004 cm. could have been detected in the sum of two double amplitudes). This gives for the momentum that could have been detected

$$\Delta M = 0.01 \text{ dyne-sec. (approximately).}^1$$

We may then say that the experiment shows that in charging a 10 mf. condenser to 240 volts, any momentum effect of the first kind postulated is less than 0.01 dyne-second.

In looking for the second kind of momentum effect, that is, an impulse in the same direction on both charge and discharge, the same general method was used. In this case, however, the condenser could be charged and discharged on alternate swings, or the sensitivity could be increased by charging and discharging several times each swing. It was found that 15 charges and discharges, or 30 operations per swing, could be easily

¹ Barrell has recently suggested the use of "second-dyne" as a name for the C.G.S. unit of momentum (*Nature*, 87, p. 144, 1911). An inversion to "dyne-second" seems decidedly better as it is more euphonious and more easily spoken.

carried out without trouble arising from heat being developed in the leading-in wires.¹

In this case also the result was negative although as before a change of 0.004 cm. in double amplitude per swing could certainly have been detected. Since 15 charges and discharges per swing were used, this means that the momentum of a charge or discharge that could have been detected was 1/30 of that in the first case, or approximately 0.0004 dyne-sec.

The results of the tests for momentum effects were independent of the position of condenser on the pan, whether upright, inverted or lying on its side.

SUMMARY.

1. Charging a 10 mf. condenser to 240 volts was found not to change its weight by as much as 0.01 mg., or one part in 100,000,000.
2. In charging and discharging the same condenser no momentum effect having opposite signs on charge and on discharge as great as 0.01 dyne-sec. was detected.
3. Similarly no momentum effect having the same sign on charge and on discharge as great as 0.0004 dyne-sec. was detected.

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¹ It may be interesting to note that a contact resistance at the condenser gave a very decided simulation of the momentum effect of the second kind by the air currents set up. Hence soldered connections were used.

A GENERAL PROBLEM IN GYROSTATIC ACTION.

By J. W. MILNOR.

THE theory of gyrostatic action has been prominent in recent technical writings due to the development of the monorail car. In the January number of the *PHYSICAL REVIEW* Professor Franklin discussed the behavior of an unconstrained gyrostat under a suddenly applied torque. The motion due to a torque proportional to the time was calculated by B. L. Newkirk in the July number. In the following paper an approximate solution in series is given for the motion caused by any applied torque whatever, where this torque is known as a function of the time.

In the practical form in which we are interested, the gyrostat consists of a wheel symmetrical about its axis, and rotating at a high rate of speed about that axis. The wheel is so mounted that its axis is free to turn in any direction. If the axis of such an arrangement be constrained to turn in a given direction, it will exert a force on the constraining mechanism at right angles to the plane in which it turns. Conversely, if an external torque acts so as to tend to change the direction of the axis of a free gyrostat, the gyrostat will resist the torque. At the same time the axis of the gyrostat will start to turn slowly about an axis at right-angles to the axis about which the disturbing torque acts, and the torque due to this precession of the axis will balance the external torque. While the precession is starting the gyrostat will yield slightly in the direction of action of the external torque.

The free gyrostat also possesses a certain degree of elasticity. It has a natural period of oscillation of its own, and its oscillations when started would continue indefinitely if undamped by friction, or otherwise. The natural period is as a rule short; if the mass of the axle and of the supporting frame of the wheel is negligible, the period is one half the time of one revolution of the wheel. If the frame is heavy, the period is increased. But unlike a perfectly elastic body, the gyrostat does not return to its original position when an external disturbing torque ceases to act on it.

The position of a gyrostat is most conveniently stated by giving its angular displacements about two perpendicular axes which are each very nearly perpendicular to the normal position of the axis of the gyrostat. In what follows it is assumed that these displacements are never large,

and the results are approximate to this extent. Any external torque acting on the gyrostat may be resolved into its components about these two axes (the third component, about the axis of the rotating wheel, not being of interest). If the reference axes be conceived to move with the frame of the gyrostat, the results may be extended to apply over large displacements of the gyrostat.

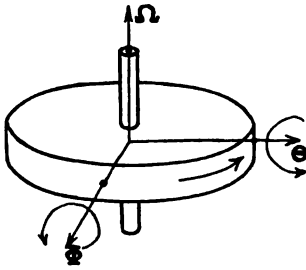


Fig. 1.

A torque $g(t)$ acting to turn the gyrostat about the Φ -axis (see Fig. 1) may be separated into two components: (1) a component which goes to accelerate the gyrostat about this axis at a rate equal to this component divided by the moment of inertia K of the wheel and mounting about this axis; (2) a component which causes precession about the Θ -axis equal to this torque-component divided by $I\omega$, where I and ω are the moment of inertia and angular velocity, respectively, of the rotating wheel. Expressed as a differential equation, this torque is

$$g(t) = K \frac{d^2\phi}{dt^2} + I\omega \frac{d\theta}{dt}. \quad (1)$$

Similarly

$$f(t) = J \frac{d^2\theta}{dt^2} - I\omega \frac{d\phi}{dt}. \quad (2)$$

Here $f(t)$ and J are respectively the torque about the Θ -axis, and the moment of inertia of the wheel and frame about the Θ -axis. The difference in sign is due to the positions of the axes relative to each other.

Differentiating (1) and (2), and combining, we have

$$JK \frac{d^3\phi}{dt^3} + I^2\omega^2 \frac{d\phi}{dt} + I\omega f(t) - Jg'(t) = 0 \quad (3)$$

and

$$JK \frac{d^3\theta}{dt^3} + I^2\omega^2 \frac{d\theta}{dt} - I\omega g(t) - Kf'(t) = 0. \quad (4)$$

Equation (3) will be solved in series by the parametric method devised by Professor P. A. Lambert.¹ In place of (3) we may write

$$\frac{d^3\phi}{dt^3} = -\frac{I^2\omega^2}{JK} \frac{d\phi}{dt} - m - \frac{I\omega}{JK} f(t) + \frac{1}{K} g'(t), \quad (5)$$

¹ Annals of Mathematics, July, 1910.

and assume

$$\phi = \phi_0 + \phi_1 m + \phi_2 m^2 + \dots \quad (6)$$

makes equation (5) an identity, where m is a parameter which is later placed equal to unity.

Differentiating (6) successively,

$$\frac{d\phi}{dt} = \frac{d\phi_0}{dt} + \frac{d\phi_1}{dt} \cdot m + \frac{d\phi_2}{dt} \cdot m^2 + \dots, \quad (7)$$

$$\frac{d^2\phi}{dt^2} = \frac{d^2\phi_0}{dt^2} + \frac{d^2\phi_1}{dt^2} \cdot m + \frac{d^2\phi_2}{dt^2} \cdot m^2 + \dots \quad (8)$$

Inserting these values of the derivatives in (5), equating like powers of m , and solving the resulting equations in order, we readily obtain:

$$\begin{aligned} \phi_0 &= \frac{a't^2}{2} + b't + c' - \frac{I\omega}{JK} f^{-2}(t) + \frac{1}{K} g^{-2}(t), \\ \phi_1 &= -\frac{I^2\omega^2 a't^4}{JK} \frac{1}{4} - \frac{I^2\omega^2 b't^3}{JK} \frac{1}{3} + \frac{I^3\omega^3}{J^2K^2} f^{-5}(t) - \frac{I^2\omega^2}{JK^2} g^{-4}(t), \\ \phi_2 &= \frac{I^4\omega^4 a't^6}{J^2K^2} \frac{1}{6} + \frac{I^4\omega^4 b't^5}{J^2K^2} \frac{1}{5} - \frac{I^5\omega^5}{J^3K^3} f^{-7}(t) + \frac{I^4\omega^4}{J^2K^3} g^{-6}(t), \\ \phi^3 &= \text{etc.} \end{aligned} \quad (9)$$

Here a' , b' , and c' are arbitrary constants, to be determined by the initial conditions; $f^{-1}(t)$ is written as a symbol for the definite integral

$$\int_0^t f(t)dt; f^{-2}(t) \text{ is a symbol for } \int_0^t \int_0^t f(t)dt^2; \text{ etc.}$$

Inserting equations (9) in (6), putting $m = 1$, and combining the a' and b' terms into cosine and sine series respectively, we obtain the general solutions,

$$\begin{aligned} \phi &= a \cos \frac{I\omega t}{\sqrt{JK}} + b \sin \frac{I\omega t}{\sqrt{JK}} + c - \frac{I\omega}{JK} \left[f^{-2}(t) - \frac{I^2\omega^2}{JK} f^{-5}(t) + \dots \right] \\ &\quad + \frac{1}{K} \left[g^{-2}(t) - \frac{I^2\omega^2}{JK} g^{-4}(t) + \dots \right] \quad (10) \end{aligned}$$

and from (1)

$$\begin{aligned} \theta &= \int \left[\frac{g(t)}{I\omega} - \frac{K}{I\omega} \frac{d^2\phi}{dt^2} \right] dt \\ &= a \sqrt{\frac{K}{J}} \sin \frac{I\omega t}{\sqrt{JK}} - b \sqrt{\frac{K}{J}} \cos \frac{I\omega t}{\sqrt{JK}} + d \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{J} \left[f^{-2}(t) - \frac{I^2 \omega^2}{JK} f^{-4}(t) + \dots \right] \\
& + \frac{I\omega}{JK} \left[g^{-3}(t) - \frac{I^2 \omega^2}{JK} g^{-5}(t) + \dots \right]. \quad (11)
\end{aligned}$$

Another particular solution of the general equations may be obtained by writing in place of equation (5),

$$\frac{d\phi}{dt} = -\frac{JK}{I^2 \omega^2} \frac{d^2 \phi}{dt^2} \cdot m - \frac{1}{I\omega} f(t) + \frac{J}{I^2 \omega^2} g'(t), \quad (12)$$

and by a precisely similar method of solution are obtained:

$$\begin{aligned}
\phi = & a \cos \frac{I\omega t}{\sqrt{JK}} + b \sin \frac{I\omega t}{\sqrt{JK}} + c \\
& - \frac{1}{I\omega} \left[f^{-1}(t) - \frac{JK}{I^2 \omega^2} f'(t) + \frac{J^2 K^2}{I^4 \omega^4} f'''(t) - \dots \right] \\
& + \frac{J}{I^2 \omega^2} \left[g(t) - \frac{JK}{I^2 \omega^2} g''(t) + \dots \right] \quad (13)
\end{aligned}$$

and

$$\begin{aligned}
\theta = & a \sqrt{\frac{K}{J}} \sin \frac{I\omega t}{\sqrt{JK}} - b \sqrt{\frac{K}{J}} \cos \frac{I\omega t}{\sqrt{JK}} + d + \frac{K}{I^2 \omega^2} \left[f(t) - \frac{JK}{I^2 \omega^2} f''(t) + \dots \right] \\
& + \frac{1}{I\omega} \left[g^{-1}(t) - \frac{JK}{I^2 \omega^2} g'(t) + \frac{J^2 K^2}{I^4 \omega^4} g'''(t) \dots \right] \quad (14)
\end{aligned}$$

in which the complementary solutions, represented by the first three terms of the right-hand members of (10) and (11), have been included.

Either set of solutions (10) and (11), or (13) and (14), holds when $f(t)$ and $g(t)$ are analytical functions of t . If the functions $f(t)$ and $g(t)$ are arbitrary functions, or are expressed as a Fourier's series, the first set alone holds; provided in any case that the resultant series terminates or is convergent.

Placing $t = 0$ in (10) and (11), we have

$$\begin{aligned}
\phi_0 &= a + c, \\
\theta_0 &= -\sqrt{\frac{K}{J}} \cdot b + d
\end{aligned} \quad (15)$$

and by differentiation,

$$\begin{aligned}
\left(\frac{d\phi}{dt} \right)_0 &= \frac{I\omega b}{\sqrt{JK}}, \\
\left(\frac{d\theta}{dt} \right)_0 &= \frac{I\omega a}{J},
\end{aligned}$$

from which a , b , c , and d are determined. Similarly, from (13) and (14),

$$\begin{aligned}\phi_0 &= a + c + \frac{JK}{I^2\omega^2} \left[f'(0) - \frac{JK}{I^2\omega^2} f'''(0) + \dots \right] \\ &\quad + \frac{J}{I^2\omega^2} \left[g(0) - \frac{JK}{I^2\omega^2} g''(0) + \dots \right], \\ \theta_0 &= -\sqrt{\frac{K}{J}} \cdot b + d + \frac{K}{I^2\omega^2} \left[f(0) - \frac{JK}{I^2\omega^2} f'''(0) + \dots \right] \\ &\quad - \frac{JK}{I^2\omega^2} \left[g'(0) - \frac{JK}{I^2\omega^2} g'''(0) + \dots \right], \\ \left(\frac{d\phi}{dt}\right)_0 &= \frac{I\omega b}{\sqrt{JK}} - \frac{1}{I\omega} \left[f(0) - \frac{JK}{I^2\omega^2} f'''(0) + \dots \right] \\ &\quad + \frac{J}{I^2\omega^2} \left[g'(0) - \frac{JK}{I^2\omega^2} g'''(0) + \dots \right], \\ \left(\frac{d\theta}{dt}\right)_0 &= \frac{I\omega a}{\sqrt{JK}} + \frac{K}{I^2\omega^2} \left[f'(0) - \frac{JK}{I^2\omega^2} f'''(0) + \dots \right] \\ &\quad + \frac{1}{I\omega} \left[g(0) - \frac{JK}{I^2\omega^2} g''(0) + \dots \right].\end{aligned}\tag{16}$$

As an example, let a torque C be applied about the Φ -axis to a stationary gyrost. Then

$$\phi_0 = 0; \quad \theta_0 = 0; \quad \left(\frac{d\phi}{dt}\right)_0 = 0; \quad \left(\frac{d\theta}{dt}\right)_0 = 0; \quad f(t) = 0; \quad g(t) = C.$$

Solving by means of equations (13), (14), and (16), we find,

$$\begin{aligned}\phi &= -\frac{JC}{I^2\omega^2} \cos \frac{I\omega t}{\sqrt{JK}} + \frac{JC}{I^2\omega^2}, \\ \theta &= -\frac{\sqrt{JK}C}{I^2\omega^2} \sin \frac{I\omega t}{\sqrt{JK}} + \frac{Ct}{I\omega},\end{aligned}\tag{17}$$

which when J is put equal to K become the results obtained by Professor Franklin.

If $g(t)$ is placed equal to kt , we readily obtain from (13), (14), and (16) the results, for a symmetrical gyrost ($J = K$),

$$\begin{aligned}\phi &= -\frac{J^2k}{I^2\omega^2} \sin \frac{I\omega t}{J} + \frac{Jkt}{I^2\omega^2}, \\ \theta &= \frac{J^2k}{I^2\omega^2} \cos \frac{I\omega t}{J} - \frac{J^2k}{I^2\omega^2} + \frac{kt^2}{2I\omega},\end{aligned}\tag{18}$$

which are those given by Mr. Newkirk.

If the applied torque about the Φ -axis is periodic, we may put

$$g(t) = C \sin \alpha t.$$

Assuming as before that the gyrostat is stationary at $t = 0$, the solution, obtained from (13), (14), and (16), is

$$\begin{aligned} \phi &= \left[\frac{JC}{I^2\omega^2} \sin \alpha t - \frac{J^2\alpha C}{I^2\omega^2} \sin \frac{I\omega t}{J} \right] \left[1 + \frac{J^2\alpha^2}{I^2\omega^2} + \frac{J^4\alpha^4}{I^4\omega^4} + \dots \right] \\ &= \frac{C}{I\omega \left(\frac{I^2\omega^2}{J^2} - \alpha^2 \right)} \cdot \left[\frac{I\omega}{J} \sin \alpha t - \alpha \sin \frac{I\omega t}{J} \right], \\ \theta &= \frac{C}{I\omega\alpha \left(\frac{I^2\omega^2}{J^2} - \alpha^2 \right)} \cdot \left[\frac{I^2\omega^2}{J^2} (1 - \cos \alpha t) - \alpha^2 \left(1 - \cos \frac{I\omega t}{J} \right) \right]. \end{aligned} \quad (19)$$

If α is put equal to $I\omega/J$ the two periods are equal, and equations (19) take the form 0/0. This may be evaluated by the rules of differential calculus, and there results,

$$\begin{aligned} \phi &= -\frac{Ct}{2I\omega} \cos \frac{I\omega t}{J} + \frac{CJ}{2I^2\omega^2} \sin \frac{I\omega t}{J}, \\ \theta &= -\frac{Ct}{2I\omega} \sin \frac{I\omega t}{J} + \frac{CJ}{I^2\omega^2} \left(1 - \cos \frac{I\omega t}{J} \right). \end{aligned} \quad (20)$$

These equations show that the amplitude of the resulting oscillations steadily increases with the time.

As a final example, consider the effect of a sudden impulse to a stationary gyrostat. Let the torque C act about the Φ -axis for the time T , T being very short. We shall use equations (10), (11), and (15) in the solution. Then $f(t) = 0$ for all values of t ; $g(t) = C$ when $t < T$, and $g(t) = 0$ when $t > T$. $g^{-1}(t) = Ct$ when $t < T$, and $g^{-1}(t) = CT$ when $t > T$.

$$g^{-2}(t) = \frac{Ct^2}{2}$$

when $t < T$, and

$$g^{-2}(t) = \frac{CT^2}{2} + CT(t - T)$$

when $t > T$. Since T is very small we may without sensible error neglect all powers of T higher than the first, and $g^{-2}(t)$ becomes CTt . Similarly,

$$g^{-3}(t) = \frac{CT^2}{2}, \text{ etc.}$$

From equations (15) we find $a = 0 = b = c = d$. Then by (10) and (11) we have by combining the resultant quantities into sine and cosine series respectively,

$$\begin{aligned}\phi &= \frac{CT}{I\omega} \sin \frac{I\omega t}{J}, \\ \theta &= \frac{CT}{I\omega} \left[1 - \cos \frac{I\omega t}{J} \right],\end{aligned}\tag{21}$$

which give the position of the gyrostat at any time t after a sudden impact CT .

ON THE NATURE OF THE VOLTA EFFECT.

BY FERNANDO SANFORD.

THE fundamental question as to the nature of the Volta effect seems to be whether the different metals when in electrical contact with the earth or with the inside of the same hollow conductor are still at different potentials relative to each other, or whether the potential difference which may be observed between two of them when they are brought very close together is due to some kind of an electrolytic polarization in the medium between them.

The theory of electrolytic polarization has generally been so stated as to require a metallic contact between the metals before an electric field is set up between them, but the discovery of the "approach charges" of Righi and Majorana made this part of the theory untenable. It has still been maintained, however, that the electric field which is observed between the two metals when they are brought close together is not due to an original potential difference between the metals themselves, but to a potential difference between liquid or gaseous layers which are assumed to be condensed upon the metallic surfaces.

Since in order to make this theory tenable it is necessary to assume the presence of these condensed layers on the metals before they are brought near together, it seems to be implied that there is normally a double electric layer over the surface of the metal, one charge being upon the metal and the other in the liquid or gaseous layer very near the metallic surface. This view would seem to require that the surfaces of the two different metals be brought very close together before the electric field may be observed between them.

It is possible, however, to observe the approach charges without bringing the opposed metal faces very close together. One way in which this may be accomplished is by lowering an insulated conductor of one metal, as zinc, into an insulated hollow conductor of another metal, as copper. When this is done, an induced charge may be taken from the outer surface of the hollow conductor. The phenomenon is easily shown by connecting one pair of quadrants of an electrometer or the leaf of a Wilson gold-leaf electroscope to the hollow metal conductor, which is first grounded and then insulated. A ball or other convenient mass of the other metal is suspended by an insulating thread, and is first grounded

and then insulated and lowered into the hollow conductor, but is not allowed to touch it. The deflection of the electrometer needle or the gold leaf shows the induced charge upon the outer surface of the hollow conductor. If the hollow conductor be now discharged to earth and the inside conductor be withdrawn, the hollow conductor is again charged, but with the opposite sign.

It seems impossible to explain this phenomenon by any assumption of a double electric layer over the surface of the inside conductor, since such a layer must consist of equal positive and negative charges, and no number of equal positive and negative charges inside a hollow conductor could induce a charge on its outer surface.

But a still more crucial experiment may be performed with the hollow conductor. On any electrolytic theory of the Volta effect, the transference of electricity between the two metal surfaces when they are brought into contact must depend upon the nature of the opposed metallic surfaces. Accordingly, if a sphere of one metal be brought into contact with the inside of a hollow conductor of another metal, the charge which the sphere will take from the hollow conductor will depend upon the metal of the *inner* surface of the hollow conductor. If, on the other hand, the different metals when in contact with the earth are at different potentials relative to each other (since the potential of the inside of a hollow conductor must be the same as the outside) the potential difference between the inclosed sphere and the outside hollow conductor will depend upon the nature of the metal on the outside of the hollow conductor. The charge which the sphere will take when touched to the inside of the hollow conductor will, from this point of view, depend upon the metal of the *outer* surface of the hollow conductor.

It is not difficult to decide between these hypotheses. I have done so in the following manner: A gold-leaf electroscope of the C. T. R. Wilson pattern was mounted upon a stand about a foot above the supporting pier. Its outer case was connected by a copper wire to the water system of the laboratory. Its inducing plate was connected to one pole of a battery of 100 dry cells, the other pole of which was connected to earth. Its sensibility was such that one dry cell between the gold leaf and earth gave a deflection of about 14 scale divisions as seen in the reading microscope.

A polished zinc ball about 5 centimeters in diameter was suspended by a silk cord which passed over a fixed pulley about two feet above the pier, so that the ball could be raised or lowered by the cord without bringing the hand near the system. The suspending cord was about one foot from the electroscope, and a fine copper wire which was connected

to the gold leaf was suspended horizontally on an insulating support so that its end would touch the zinc ball when it was at the proper height.

Two beakers of about 750 c.c. capacity (10 cm. high and 10 cm. across the top), one of copper and one of aluminum, were used as hollow conductors. When in use, a beaker was placed upon the wooden top of the pier between three small brass screws, which served to fix its position, so that the zinc ball when lowered into it would touch the center of the bottom. A copper wire was laid between the screws so that the bottom of the beaker would rest upon it, and one end of this wire was soldered to a waterpipe in the laboratory. This insured that the beaker should be perfectly grounded during the experiment.

The zinc ball was then lowered into the beaker by means of the silk cord and pulley until it touched the bottom of the beaker, and was then drawn up until it touched the wire from the electroscope. As soon as the ball was outside the beaker it had a free charge which it shared with the gold leaf of the electroscope when the two were in contact. On account of the small capacity of the gold leaf and connecting wire, a very few repetitions of this process would bring the gold leaf to practically the potential of the zinc ball when it left the beaker. A series of readings were then made by successively lowering the ball to the bottom of the beaker and raising it until it touched the charging wire of the electroscope, then lowering the ball and reading the electroscope.

These readings were not constant, but varied among themselves by several scale divisions. This was probably due in part to the uncertain contact which the charging wire of the electroscope made with the zinc ball, but no special attempt was made to get rid of the various disturbing causes which affect electrostatic experiments in all laboratories. The means of several series of ten readings were found to be nearly enough constant for the purposes of this investigation, and were accordingly used.

Copper Beaker.	Aluminum Beaker.	Copper Beaker.	Aluminum Beaker.
16	19	27	23
16	29	25	28
16	35	16	31
18	37	15	32
11	40	17	34
11	42	15	40
11	44	10	37
[10	39	10	40
10	35	10	44
10	32	11	40
Mean . . . 12.9	35.2	15.6	34.9

In determining the difference between the charges induced by the two beakers, the method adopted was to make ten successive readings with one beaker, then, while the zinc ball was raised the other beaker was substituted and ten readings were made with it, when the beakers were again changed. The following is a sample series made by using the zinc ball with the aluminum and copper beakers. The series was made without grounding the electroscope or making any change in the apparatus except to substitute one beaker for the other at the end of ten readings.

The mean of the two sets from the copper beaker and the two sets from the aluminum beaker differed by 20.8 scale divisions. The electroscope at that time gave about 14 scale divisions for one dry cell, or approximately 10 scale divisions for one volt. Of course, this difference of 20.8 scale divisions does not represent the true Volta potential difference between the aluminum and the copper beakers, as whenever the beakers were exchanged several charges of the ball were required to bring the electroscope charge up to a constant value.

Two circular discs of tin foil ten centimeters in diameter which had been cut from the same sheet were then pressed down into the beakers, so that they covered the bottoms and were turned up one centimeter around the edges. Series of readings were then taken with the two beakers as before. In this case, the zinc ball touched only the center of the tin-foil disc, and if its charge were due to an electrolytic action between it and the tin foil the charge should have been the same in both beakers. The following series of readings were taken from the two beakers with the tin foil in the bottom:

Tin Foil in Aluminum.	Tin Foil in Copper.	Tin Foil in Aluminum.	Tin Foil in Copper.
44	31	34	20
45	28	31	29
47	22	31	23
45	22	35	20
39	17	33	25
42	15	37	18
41	18	43	15
41	22	45	14
41	25	47	13
38	20	49	13
Mean . . . 42.3	22	38.5	19

The means of these four series give a difference between the charges taken from the tin foil in the aluminum beaker and those taken from a disc of the same tin foil in the copper beaker of 20 scale divisions, virtually the same difference which was observed between the two beakers them-

selves. This experiment was repeated several times with similar results. Sometimes the difference between the charges taken from the tin foil was a little greater than between those taken from the naked beaker, and sometimes a little less. The tin-foil discs were interchanged without affecting the result.

The beakers were then inverted and two similar discs of tin foil were placed upon the *outside* of the bottoms. The beakers were then placed in their former positions inverted, and the zinc ball was then charged from the center of the tin-foil discs on the outside of the beakers. The following series of readings were made in this way. They differed from the other series only in that 15 readings were taken from each beaker and the first five rejected. This was because of the fact that the ball would take a much smaller charge from the outside of the beaker, and if there were a difference in the potentials of the two tin-foil discs it would take more charges to show it. The result would, however, have been scarcely changed by taking the mean of the 15.

Tin Foil on Copper.		Tin Foil on Aluminum.	
	35		38
	35		40
	37		42
	34		36
	36		36
	39		36
	35		32
	31		34
	35		35
	36		36
Mean . . .	35.3		36.5

The difference of 1.2 scale divisions between the means of these two series is not greater than the probable error of the experiment.

Beakers Covered with Tin Foil.		Beakers Uncovered.	
Aluminum.	Copper.	Copper.	Aluminum.
51	40	26	46
52	46	23	45
50	44	26	51
55	39	28	50
54	45	26	52
57	43	25	55
52	45	32	56
51	39	31	53
46	46	31	49
41	48	29	54
Mean . . .	50.9	43.4	51.1

The experiment was then varied by wrapping the outside of the beakers with tin foil, leaving an opening in the top of about 7 cm. in diameter. The beakers were placed upright and the copper ball charged from the inside. Ten readings were taken from each beaker, as before. The tin foil was then removed and ten readings taken again with nothing changed but the tin foil. The following series is the result.

In this experiment, though the beakers were not wholly enclosed in the tin foil the charges taken from their interior surfaces differed by only one third as much as when the tin foil was removed.

These experiments were in part repeated with a gilt ball instead of the zinc ball, but without changing the result. As said before, no claim is made that the actual potential difference between the beakers was obtained with any high degree of accuracy in any case. The different series of readings are, however, fairly comparable, and only one conclusion seems possible from them. Apparently the Volta effect is not due to any electrolytic action between the opposed metals, but different metals when in contact with the earth or with the inside of the same hollow conductor may be at different potentials relative to each other.

STANFORD UNIVERSITY,
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NEW BOOKS.

- Physik in Graphischen Darstellungen.* By FELIX AUERBACH. Berlin: B. G. Teubner, 1912. Pp. x + 28.
- Spectroscopy.* By E. C. C. BALY. New York: Longmans, Green and Co., 1912. Pp. xiv + 687.
- Studies in Radioactivity.* By W. H. BRAGG. New York: The Macmillan Company, 1912. Pp. x + 196. Price, \$1.60.
- Electromagnetic Radiation and the Mechanical Reactions Arising from it.* By G. A. SCHOTT. Cambridge: Cambridge University Press, 1912. Pp. xxii + 330. Price, \$5.75.
- Scientific Papers.* By JOHN WILLIAM STRUTT. Vol. V. Cambridge: Cambridge University Press, 1912. Pp. xii + 624. Price, \$5.00.
- Gravitation.* By FRANK HARRIS. New York: Longmans, Green and Co., 1912. Pp. xi + 107. Price, \$1.00.
- Les Anaglyphes Geometriques.* By H. VUIBERT. Paris: Librairie Vuibert, 1912. Pp. 1 + 32.
- Conférences sur Quelques Thèmes Choisis de la Chimie Physique pure et Appliquée.* By SVANTE ARRHÉNIUS. Paris: Librairie Scientifique A. Hermann and Fils, 1912. Pp. 1 + 112. Price, 3 Fr.
- La Théorie des Ions et l'Electrolyse.* 2d edition. By AUGUSTE HOLLARD. Paris: Gauthier-Villars, Imprimeur-Libraire, 1912. Pp. vii + 220. Price, 5 Fr.
- Wissenschaftliche Abhandlungen der Kaiserlichen Normal-Eichungskommission.* Vol. VIII. Berlin: Julius Springer, 1912. Price, 9 M.
- Studies in Light Production.* By R. A. HOUSTON. New York: D. Van Nostrand Company, 1912. Pp. 1 + 111. Price, \$2.00.

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